Relaxation and Oscillation Model Using Caputo Fractional Differential Equations

Wildy Ardan[1,a)], Siti Fatimah[2,b)], Kartika Yulianti[3,c)]

[1,2,3] Department of Mathematics Education, FPMIPA, Indonesia University of Education

[a)]E-mail:Bakawilde@upi.edu
[b)]E-mail:sitifatimah@upi.edu
[c)]E-mail:kartika.yulianti@upi.edu

Abstract

The phenomenon of relaxation and oscillation is a common event that is often encountered. Both of these properties can occur in viscoelastic materials even though they do not occur simultaneously. Because the characteristics of viscoelastic materials are difficult to describe using classical-order differential equations, in this study, fractional-order differential equations were used to model each of the relaxation and oscillation phenomena in viscoelastic materials with the help of Laplace transform as a solution method . The solution obtained characterizes the phenomenon of memory effect as well as viscoelastic materials in general. In addition to this phenomenon, several other variables were also found to be the influence of the related material motion dynamics.

Keywords*: Caputo fractional derivative, fractional derivative, oscillation, relaxation, viscoelastic.*

Introduction

In physics, relaxation is a phenomenon in which a system that has been disturbed returns to equilibrium [1]. This phenomenon is common in elastic materials. Another phenomenon is oscillation which is defined as the back and forth motion of an object that occurs periodically or the repeated motion of the object at a fixed time interval [2]. In contrast to relaxation, oscillations are not only found in mechanical systems such as springs, but also in dynamic systems. There are materials that can experience relaxation and oscillation phenomena known as viscoelastic materials. Besides being able to experience two different phenomena, this material also has physical behavior such as elastic or viscous material [3].

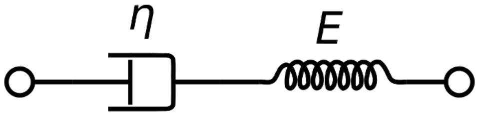

Figure 1.1 Illustration of a system with elastic and viscous components [4].

The oscillatory movement of a viscoelastic material is influenced by several things, including the nature of the material, the type of material, and the mass. However, there are some behaviors of viscoelastic materials that are quite difficult to describe through equations with an integer order, namely the memory effect behavior which is characterized by the phenomenon of slowed relaxation, or damped oscillations. This behavior only occurs in soft viscoelastic materials such as polymers, biopolymers, metals at very high temperatures, and bituminous materials. Because this behavior is difficult to describe using an integer-order derivative equation model, a fractional-order derivative equation model is used to describe this behavior [5].

Fractional-order differential equations commonly used today include; (i) Grünwald Letnikov Type, (ii) Riemann-Liouville Type, and (iii) Caputo Type [6]. The Riemann-Liouville type fractional order differential requires that the initial conditions are fractional order, so it has a physical meaning that is quite difficult to explain [6]. Meanwhile, the initial conditions specified for the Caputo-type fractional-order differential are not different from the integer-order differentials, so they can be interpreted physically. Fractional order differential equations are commonly used to accurately model systems that require accurate damping modeling [7]. In addition, Caputo fractional derivative is useful in modeling physical phenomena that have memory [8]. Based on these reasons, the Caputo fractional order derivative is considered more suitable for modeling and solving this problems.

The use of Caputo fractional derivative has been applied in solving other physical problems, including in electric circuits [9], and the semi-infinite cooling process by radiation [10]. The solution of the relaxation and oscillation model using the Caputo fractional derivative has previously been studied by Chen [5]. In this study, a mathematical model of relaxation and oscillation movements is described using the Caputo type fractional differential. The model used has a correspondence with the Maxwell model with modifications to Newton's fluid components which are replaced with soft viscoelastic components to cause memory effect behavior on the system. Then, the model is solved analytically using the Laplace transform method, which this analytical solution was not described in detail in Chen [5]. Furthermore, the relaxation and oscillation movement models were simulated to study the effect of parameters for the dynamics of relaxation and oscillation motion in viscoelastic materials.

Methods

1. Caputo Fractional Derivative

The Caputo type fractional derivative is defined as follows [11] :

$$ {}_{a}D_{x}^{\alpha}f(x) = \frac{1}{\Gamma(n-\alpha)}\int_{a}^{x}(x-t)^{n-\alpha-1}\left[\left(\frac{d}{dt}\right)^{n}f(t)\right]dt\,,\;\;(n-1<\alpha<n) \tag{1}$$

where α is a fractional order, and Γ is a gamma function. The initial condition value of the Caputo type which is not different from the value of the ordinary differential initial condition makes this type a distinct advantage compared to the Riemann-Liouville type whose initial condition value has a fractional order, so that the initial condition value can have a clear physical interpretation.

In solving Caputo fractional differential equations, the following property is used:

$$\mathcal{L}\left[\frac{d^{\alpha}}{d^{\alpha}t}f(t)\right] = s^{\alpha}F(s) - \sum_{k=1}^{m}s^{a-k}f^{k-1}(0) \tag{2}$$

2. Relaxation and Oscillation Model

2.1. Standard Relaxation Model

The standard relaxation equation is defined as follows [5]:

$$\frac{dx}{dt} + Ax = f(t), \tag{3}$$

where x represents the strain rate, $A = \frac{k}{\eta}$, k is the modulo of elasticity, η the coefficient of viscosity, and $f(t)$ is the multiply product of A and the strain rate. When $f(t) = 0$, the analytical solution is:

$$x(t) = C \exp(-At).$$

However when $f(t) \neq 0$, the analytical solution is:

$$x(t) + non - homogenous\ solution$$

with C is a constant determined by the initial conditions.

2.2. Standard Oscillation Model

The standard oscillation equation is defined as follows:

$$\frac{d^2y}{dt^2} + By = 0, \tag{4}$$

where y is the position, B equal to $\frac{K}{m} = \omega^2$, K is the spring constant, m is the mass, and ω is the angular velocity. The analytical solution of equation (4) is:

$$y(t) = C \cos\sqrt{B}t + D\, sin\sqrt{B}t,$$

where C and D are constants determined from the initial conditions.

3. Mittag-Leffler function

A natural function that arises when solving a fractional differential equation is:

$$E_\alpha(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, (\alpha \in C, Re(\alpha) > 0). \tag{5}$$

In solving the fractional differential equation, the following property is used:

$$\mathcal{L}\left[x^{\beta-1} E_{\alpha,\beta}(\lambda x^\alpha)\right] = \frac{s^{\alpha-\beta}}{s^\alpha - \lambda}. \tag{6}$$

3.1. Fractional Relaxation Model

The fractional relaxation equation is defined as follows [5]:

$$\frac{d^p x}{dt^p} + Ax = 0,\ 0 < p < 1 \tag{7}$$

where x is the magnitude of the strain in the system, A expressed as $\frac{k}{\eta}$, k is the modulo of elasticity, and η is the coefficient of viscosity.

Using the Laplace transform and initial value $x(0) = 1$, the analytical solution of Equation (7) is:

$$\mathcal{L}\{\,{}_0D_t^p x(t) + Ax(t)\} = 0 \Leftrightarrow \mathcal{L}\{\,{}_0D_t^p x(t)\} + A\mathcal{L}\{x(t)\} = 0$$

$$\Leftrightarrow \left\{ s^p X(s) - \sum_{k=0}^{n-1} s^{p-k-1} x^k(0) \right\} + AX(s) = 0$$

$$\Leftrightarrow s^p X(s) - \sum_{k=0}^{0} s^{p-k-1} x^k(0) + AX(s) = 0$$

$$\Leftrightarrow s^p X(s) - s^{p-1} x(0) + AX(s) = 0$$

$$\Leftrightarrow (s^p + A)X(s) - s^{(p-1)}x(0) = 0$$
$$\Leftrightarrow X(s) = x(0)\frac{s^{p-1}}{s^p + A}$$
$$\Leftrightarrow X(s) = x(0)\mathcal{L}^{-1}\left\{\frac{s^{p-1}}{s^p + A}\right\}$$

Using the Mittag-Leffler function, we get the inverse of the Laplace transform:

$$x(t) = x(0)\{t^0 E_{\alpha,\beta}(\lambda t^\alpha)\}$$
$$\Leftrightarrow x(t) = x(0)E_{p,1}(-At^p)$$
$$\Leftrightarrow x(t) = x(0)\sum_{k=0}^{\infty}\frac{(-At^p)^k}{\Gamma(pk+1)} \tag{8}$$

3.2. Fractional Oscillation Model

The fractional oscillation equation is defined as follows:

$$\frac{d^p y}{dt^p} + By = 0,\ 1 < p < 2 \tag{9}$$

where y is the position of the spring, B equals to $\frac{k}{m} = \omega^2$, k is the spring constant, m is the mass of the object, ω is the angular velocity.

Using the Laplace transform and initial values $y(0) = 1, y^{'}(0) = 0$, the analytical solution of equation (9) is as follows:

$$\{\,{}_0D_t^p y(t) + By(t)\} = 0 \Leftrightarrow \mathcal{L}\{\,{}_0D_t^p y(t)\} + B\mathcal{L}\{y(t)\} = 0$$
$$\Leftrightarrow \left\{s^pY(s) - \sum_{k=0}^{n-1} s^{p-k-1}y^k(0)\right\} + BY(s) = 0$$
$$\Leftrightarrow s^pY(s) - \sum_{k=0}^{1} s^{p-k-1}y^k(0) + BY(s) = 0$$
$$\Leftrightarrow s^pY(s) - s^{p-1}y(0) - s^{(p-2)}y^{'}(0) + BY(s) = 0$$
$$\Leftrightarrow (s^p + B)Y(s) - s^{(p-1)}y(0) - s^{(p-2)}y^{'(0)} = 0$$
$$\Leftrightarrow Y(s) = \frac{y(0)S^{p-1} + y^{'}(0)s^{p-2}}{s^p + B}$$
$$\Leftrightarrow Y(s) = y(0)\mathcal{L}^{-1}\left\{\frac{s^{p-1}}{s^p + B}\right\} + y^{'(0)}\mathcal{L}^{-1}\left\{\frac{s^{p-2}}{s^p + B}\right\}$$

The inverse of the Laplace transform is:

$$y(t) = y(0)E_{p,1}(-Bt^p) + y^{'}(0)E_{p,2}(-Bt^p)$$
$$y(t) = y(0)\sum_{k=0}^{\infty}\frac{(-Bt^p)^k}{\Gamma(pk+1)} + y'(0)\sum_{k=0}^{\infty}\frac{(-Bt^p)^k}{\Gamma(pk+2)} \tag{10}$$

4. Simulation

4.1. Effect of the order values in fractional relaxation equation

The following is a simulation of the solution of equation (8) for the parameter $A = 1; x(0) = 1;$ and various fractional order values compared to the standard equation $(p = 1)$.

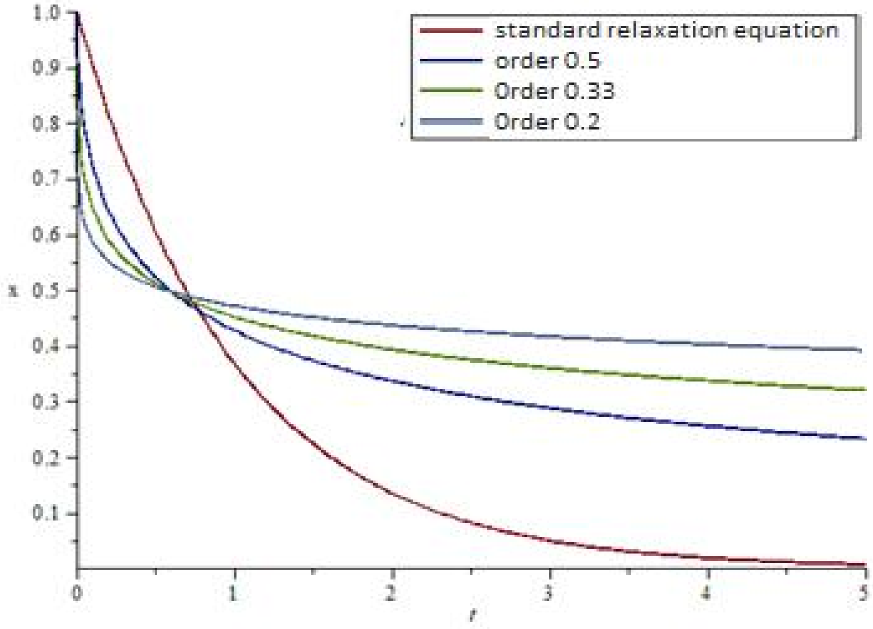

Figure 2. Graphics of Relaxation Equations

Based on Figure 2, it can be seen that there is a significant difference between the behavior of the standard relaxation equation and the fractional-order relaxation equation, namely in the standard relaxation equation the state of the system will return to a state of equilibrium with the magnitude of the stress on the system being zero, while in the fractional relaxation equation the system reaches at a state of equilibrium with the value of the voltage is not equal to zero. Figure 2 illustrates that in equation (7) the phenomenon of stress relaxation occurs which is one indicator of the memory effect. In addition, the graphs of the three fractional-order relaxation equations show quite similar behavior to each other. It is just that if the fractional order moves closer to 1 then the equilibrium state of the system will move closer to the t axis $(x = 0)$.

4.2. Effect of the order values in fractional oscillation equation

The simulation of the solution of equation (10) for the parameter values $B = 1; y(0) = 1; y^{'}(0) = 1;$ and some fractional order values is compared with the standard equation $(p = 2)$ is displayed in Figure 3.

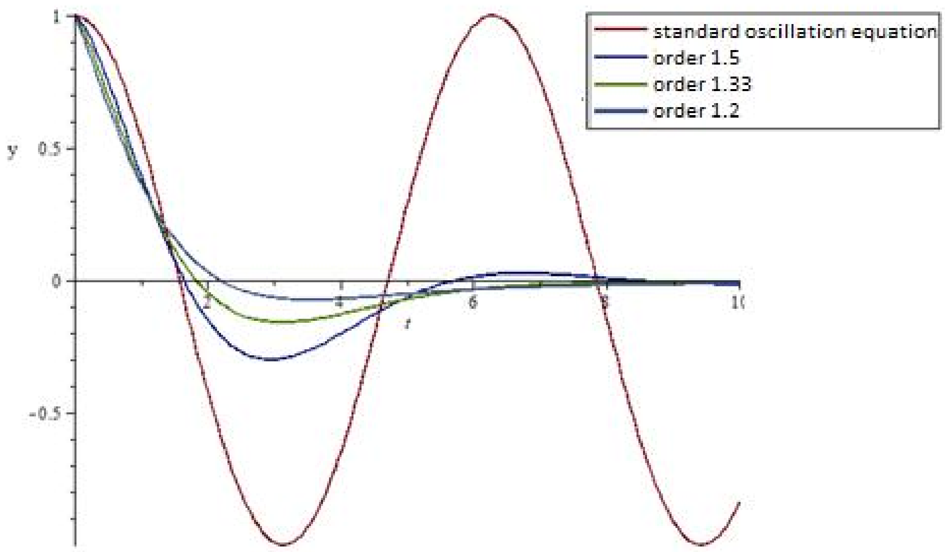

Figure 3. Graphic of Oscillation Equation solution

Based on Figure 3, it can be seen that there is a significant difference between the behavior of the standard oscillation equation and the fractional order relaxation equation, namely in the standard oscillation equation the system experiences periodic oscillations and has a constant amplitude. While in the fractional oscillation equation, the system experiences oscillations with a decreasing/shrinking amplitude value and then towards to zero. In other words, for the long time, the system returns to equilibrium. This behavior illustrates that there is a certain force that makes the spring unable to oscillate harmoniously, or in other words, the spring experiences damping when it oscillates. Thus, it can be said that the system in the fractional oscillation equation experiences a damped oscillation phenomenon which is also an indicator of the memory effect.

4.3. Effect of Modulo of Elasticity and Coefficient of Viscosity

Solution simulation with fractional order value $p = 0.5$, and some values A from equation (8) are as follows :

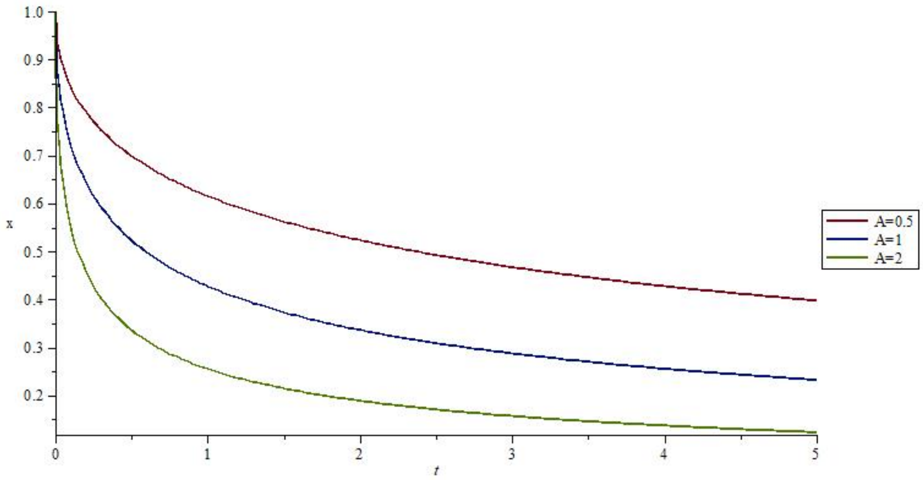

Figure 4. Graph of the Solution of the Relaxation Equation For Some values A

Based on the graph in Figure 4, information is obtained that the greater the value of A, the equilibrium state tends to resemble the standard equation or approach the t-axis $(x = 0)$, and vice versa. This is due to the fact that the tensile strength of the system also depends on the size of the modulus of elasticity and the coefficient of viscosity, or in other words, the larger the modulus of elasticity, the worse the tensile strength.

4.4. Effect of Spring Constant, Mass, and Angular Velocity

Using different B values and fractional order p = 1.5, the simulation solution of equation (10) is as follows:

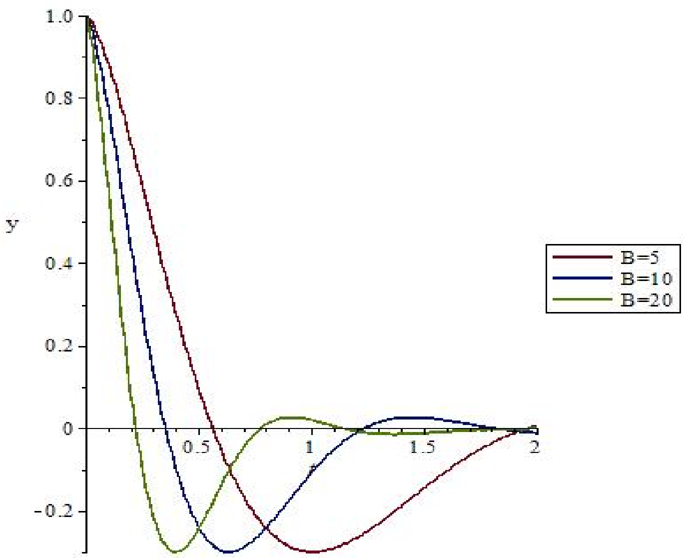

Figure 5. Graph of the Solution of the Oscillation Equation for Multiple B Values

Based on Figure 5, the oscillatory behavior shown by these three B values is quite similar to each other, only the oscillations will balance faster if the B value is also large. That is, the greater the spring constant or the smaller the mass of the object, the faster the oscillations will balance. This is in accordance with Brown's research [12] which states that the amount of attenuation is related to the mass of the object, or in other words, the greater the mass of the object, the smaller the damping power and vice versa.

Conclusion

In this study, the dynamics of the relaxation-oscillation movement of a viscoelastic object has been modeled using a Caputo type fractional differential equation. The analytical solution of the model is obtained by the Laplace transform method. Parameters that affect the dynamics of motion from the relaxation and oscillation equations include; order of fractional, modulo of elasticity, coefficient of viscosity, spring constant, mass of object, and angular velocity. For the relaxation equation, the fractional order affects the movement of the equilibrium state where if the fractional order moves closer to 1 then the equilibrium state of the system will move closer to the t axis, but the modulo of elasticity and viscosity affect the tensile strength of the system, where the larger the modulo of viscosity, the greater the modulo of viscosity. the tensile strength is getting better.

As for the oscillation equation, the fractional order followed by the spring constant, the mass of the object, and the angular velocity affect the damping speed of the system, where the greater the fractional order, the spring constant, or the smaller the mass of the object, the faster the oscillation motion will balance.

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