On The Edge Irregularity Strength of Firecracker Graphs $F_{2,m}$

Rismawati Ramdani[1, a)] and Desi Laswati Suwandi[1), b)]

[1]*Department of Mathematics, UIN Sunan Gunung Djati Bandung, Indonesia*

[a)]*email: rismawatiramdani@uinsgd.ac.id*
[b)]*email: Desilaswati28@gmail.com*

Abstract

Let $G = (V, E)$ be a graph and k be a positive integer. A vertex k-labeling $f : V(G) \rightarrow \{1,2,\cdots,k\}$ is called an edge irregular labeling if there are no two edges with the same weight, where the weight of an edge *uv* is $f(u) + f(v)$. The edge irregularity strength of *G*, denoted by *es*(*G*), is the minimum *k* such that G has an edge irregular *k*-labeling. This labeling was introduced by Ahmad, Al-Mushayt, and Bacˇa in 2014. An (n,k)-firecracker is a graph obtained by the concatenation of n k-stars by linking one leaf from each. In this paper, we determine the edge irregularity strength of fireworks graphs $F_{2,m}$.

Keywords: edge irregular labeling, firecracker, the edge irregularity strength

Introduction

Graph labeling was first introduced by Sadlàčk (1964), then Stewart (1966), Kotzig and Rosa (1970). The process of labeling a graph includes assigning values (labels), represented by a set of positive integers to vertices, edges, or both. These numbers are called labels [1]. There are several types of labeling on graphs, including gracefull labeling, harmony labeling, total irregular labeling, magic labeling, and anti-magic labeling. The concept of irregular labeling on a graph was first introduced by Chartrand et al. in 1986 [2].

In 2014, Ahmad et al. [3] introduced edge irregular labeling of graphs, namely edge irregular labeling. For an integer k, a total labeling $f : V(G) \rightarrow \{1, 2, \cdots k\}$ is called an edge irregular k-labeling of G if every two distinct edges e_2 and e_2 in E satisfy $w_f(e_1) \neq w_f(e_2)$, where $w_f(e_1 = uv) = f(u) + f(v)$. As an example, we have a graph $P_2 \cup C_3$ in the Figure 1.

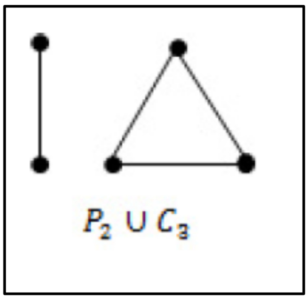

Figure 1. Given a graph $P_2 \cup C_3$

Then, we give a vertex labeling of the graph. The labeling is can be seen in the Figure 2.

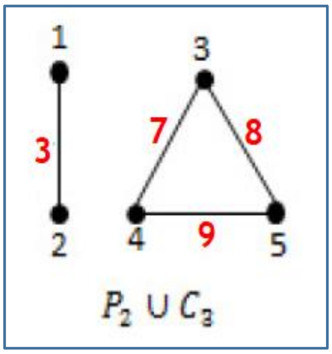

Figure 2. An edge irregular 5-labeling of $P_2 \cup C_3$

By the labeling in the Figure 2, the weight of edges of the graph are 3, 7, 8, and 9 and the maximum label used in this labeling is 5. Besides that, there are no two edges with the same weight, so the labeling is an edge irregular k-labeling of $P_2 \cup C_3$ with k=5.

The minimum k for which a graph G has an edge irregular k-labeling, denoted by $es(G)$, is called *the edge irregularity strength* of G. For example, given an edge irregular 3-labeling of $P_2 \cup C_3$ in the Figure 3.

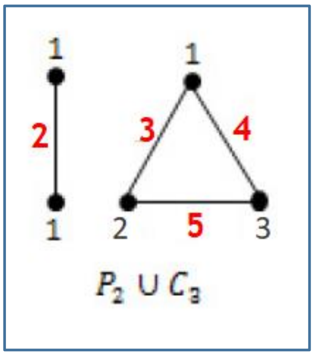

Figure 3. An edge irregular 3-labeling of $P_2 \cup C_3$

The labeling of the Figure 3 is in edge irregular 3-labeling of $P_2 \cup C_3$ because there are no two edges with the same weight and the maximum label used is 3. It is impossible to have an edge irregular k-labeling of $P_2 \cup C_3$ with maximum label 2. So, 3 is the minium k for which $P_2 \cup C_3$ has an edge irregular k-labeling. We conclude that the edge irregularity strength of $P_2 \cup C_3$ is 3, denoted by **es($P_2 \cup C_3$)=3.**

To get the exact value of es of a graph *G*, we would previously determine a lower bound and an upper bound of *es*(*G*) before. A lower bound on *es*(*G*) is obtained by using the theorem from Martin Baca and Ali Ahmad in 2014. [3] as follows.

Theorem 1 [3] : Let $G = (V, E)$ be a graph with the maximum degree Δ, then

$$es(G) \geq maks\left\{\left\lceil\frac{|E(G)|+1}{2}\right\rceil, \Delta(G)\right\}$$

Other results about computing the edge irregularity strength of graphs are given by Imran et al. in [4]. In the paper, Imran et al. determined the egde irregularity strength of caterpillars, n-star graphs, kite graphs, cycle chains and friendship graphs.

Tarawneh et al. [5], determined he edge irregularity strength of corona product of cycle with isolated vertices. In [6], Tarawneh et al. determined he exact value of edge irregularity strength for triangular grid graph , zigzag graph and Cartesian product $P_n x P_m x P_2$.

In 2017, Ahmad et al determined the edge irregularity strengths of some chain graphs and the join of two graphs. They also introduced a conjecture and open problems for researchers for further research [7]. Ahmad et al. [6], gave computing of the edge irregularity strength of bipartite graphs and wheel related graphs. In [8] , Asim et al. gained an edge irregular k-labeling for several classes of trees. Asim et al also gained the edge irregularity strength of disjoint union of star graph and subdivision of star graph [9].

In 2020, Ahmad et al performed a computer based experiment dealing with the edge irregularity strength of complete bipartite graphs. They also presented some bounds on this parameter for wheel related graphs [6]. In [10], Tarawneh et al. gave the edge irregularity strength of some classes of plane graphs.

In this paper, we determined the exact value of es of firecracker graphs $F_{2,m}$ with arbitrary m.

A firecracker is a graph obtained by the concatenation of stars by linking one of leaf from each. If the number of stars is n and the number of leaves in each star is m, then the firecracker is denoted by $F_{n,m}$.

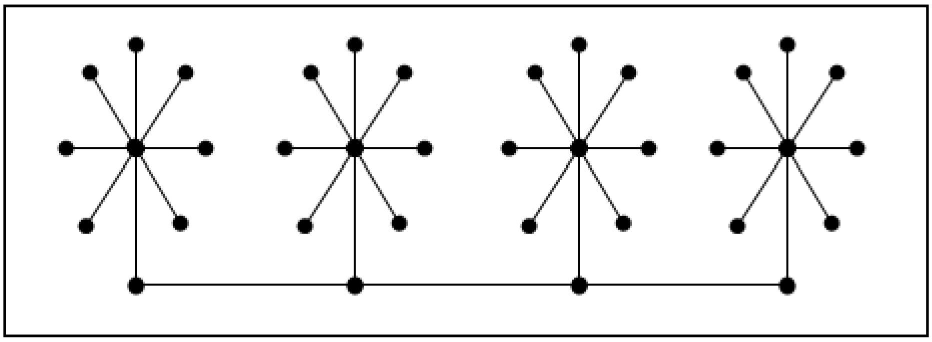

Figure 4. Firecracker graph $F_{4,8}$

Methods

The method we use in this research is analytical method. To get the exact value of es of firecracker graph, we consider a lower bound and an upper bound of $es(F_{n,m})$. Theorem 1 is used to have a lower bound of $es(F_{n,m})$. Besides that, an upper bound of $es(F_{n,m})$, we construct an edge irregular-k labeling with minimum k.

Result and Discussion

The main result of our research is the edge irregularity strength of firecracker graphs $\boldsymbol{F_{2,m}}$ is $\boldsymbol{m+1}$. The result written in Theorem 2.

Theorem 2 : Let firecracker graphs $\boldsymbol{F_{2,m}}$, for $m \geq 2$, we have edge irregularity strength by

$$es(F_{2,m}) = m+1$$

Proof.

We consider

$$es(F_{2,m}) \geq m+1 \tag{1}$$

and

$$es(F_{2,m}) \leq m+1 \tag{2}$$

To prove inequality (1), we use Theorem 1.

In the Figure 5, we can see an illustration of firecracker graph $F_{2,m}$

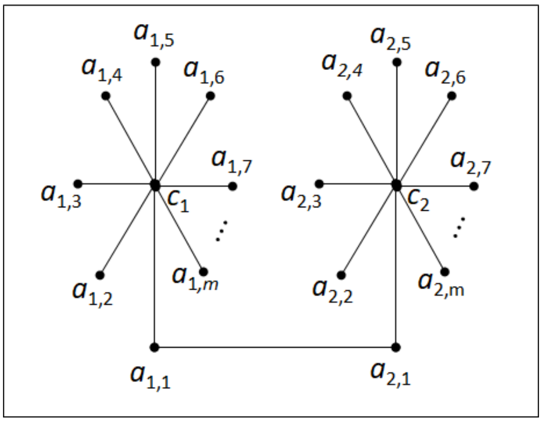

Figure 5. Firecracker graph $F_{2,m}$

From the illustration of Figure 5, we have the graph $F_{2,m}$ has $2m+1$ edges and the maximum degree $\Delta = m$. By using Theorem 1, we have

$$es(F_{2,m}) \geq \left\{\lceil\frac{|E(F_{2,m})|+1}{2}\rceil, \Delta(F_{2,m})\right\} = maks\left\{\lceil\frac{2m+2}{2}\rceil, m\right\} = maks\{m+1, m\} = m+1.$$

So, we have

$$es(F_{2,m}) \geq m+1$$

Next, we give an edge irregular k −labeling with $k = m+1$ to get $es(F_{2,m}) \leq m+1$ as follows.

For $m \geq 2$.

$$f(c_i) = \begin{cases} 1 & for\ i=1 \\ m+1 & for\ i=2 \end{cases}$$

$$f(a_{i,j}) = \begin{cases} f(c_i), & for\ j=1 \\ j & for\ i=1\ and\ 1 < j \leq m \\ f(a_{1,1}) + (j-1), & for\ i=2\ and\ 1 < j \leq m \end{cases} \tag{3}$$

From the labeling formula (3), we have the weight of edges of firecracker graphs $F_{2,m}$ as follows :

$$w_f(c_i a_{i,j}) = \begin{cases} j+1, & for\ i = 1\ and\ 1 \leq j \leq m\ ; \\ mi+i, & for\ i = 2\ and\ j = 1\ ; \\ mi+i-(m+1)+j & for\ i = 2\ and\ 1 < j \leq m\ ; \end{cases} \quad (4)$$
$$w_f(a_{i,1} a_{i+1,1}) = \begin{cases} m+2, & for\ i = 1\ ; \\ 2m+1, & for\ i = 2. \end{cases}$$

From the edge weight formula (4), there are no two edges with the same weight. The maximum label used in the labeling f is $m+1$. So, f is an edge irregular-$(m+1)$ of $F_{n,m}$. So, we can conclude that

$$es(F_{2,m}) \leq m+1$$

From inequalities (1) and (2), we have an equality

$$es(F_{2,m}) = m+1.$$

For an illustration, in the Figure 6, we can see the edge irregular labeling f of firecracker graph $F_{2,m}$ $m = 8$.

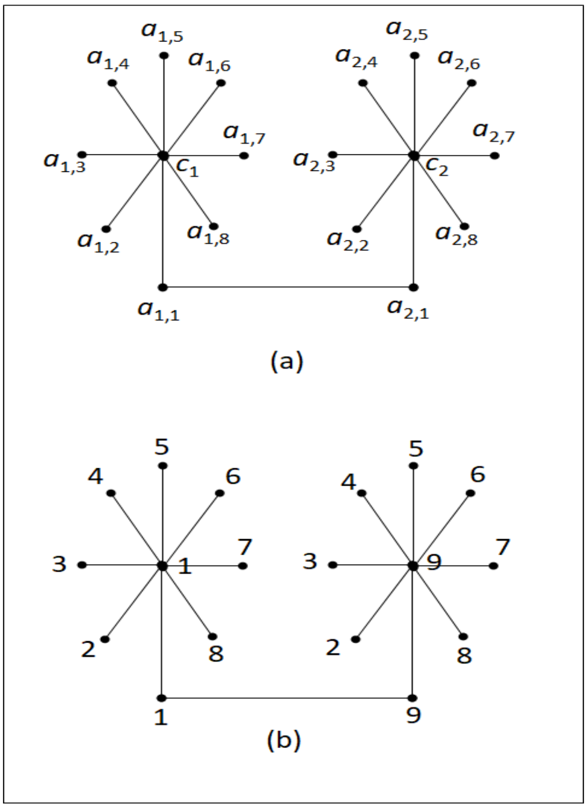

Figure 6. (a) An illustration of $F_{2,8}$; (b) An illustration of edge irregular-9 labeling f of $F_{2,8}$

In the Figure 7, we can see the weight of edges of $F_{2,8}$ under the labeling in Figure 6.

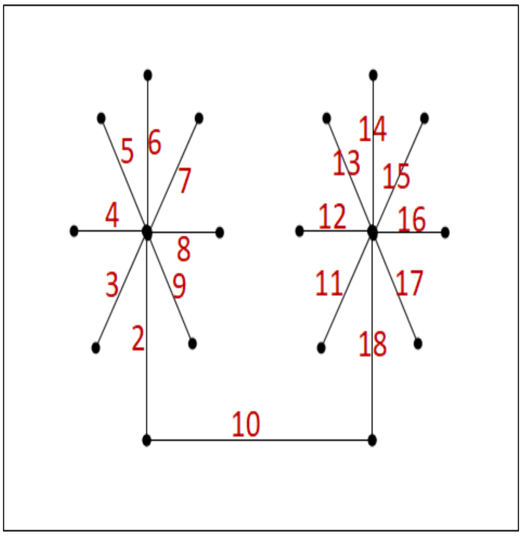

Figure 7. The weight of edges of $F_{2,8}$ under the labeling f

From the illustration of the Figure 7, we can see that there are no two edges in $F_{2,8}$ with the same weight under the labeling f.

Conclusion

By the research, we have a lower bound of $es(F_{2,m})$ is $m+1$, which is also an upper bound of $es(F_{2,m})$. So that, we can conclude the exact value of the edge irregularity strength of firecracker graphs $F_{2,m}$ is $m+1$ for $m \geq 2$.

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