Inverse RSLPFL*circfr* $\left(0,\frac{1}{b},0\right)$ Matrix of Order 3×3 to the Power of Positive Integer Using Adjoin Method

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Abstract

The *RSLPFLcircfr* $\left(0,\frac{1}{b},0\right)$ matrix is a particular form of the circular *RSLPFLcircfr*$(a_0,a_1,\dots,a_{n-1})$ matrix. This study aims to determine the general form of the inverse *RSLPFLcircfr* $\left(0,\frac{1}{b},0\right)$ matrix to the power of positive integers. This research begins by determining the general form of the power of the *RSLPFLcircfr* $\left(0,\frac{1}{b},0\right)$ matrix which is then proven by using mathematical induction. Next, predicting the determinant of the power of the *RSLPFLcircfr* $\left(0,\frac{1}{b},0\right)$ matrix which is then continued by proving the form generalization of the determinant of the power of the *RSLPFLcircfr* $\left(0,\frac{1}{b},0\right)$ matrix by direct proof using cofactor expansion. Furthermore, by determining the cofactor matrix of the power of the *RSLPFLcircfr* $\left(0,\frac{1}{b},0\right)$ matrix we will obtain results the inverse of the matrix to the power of the *RSLPFLcircfr* $\left(0,\frac{1}{b},0\right)$ matrix using the adjoin method of Equation (5).

Keywords: *Inverse; Adjoin Methods; RSLPFLcircfr.*

Introduction

A circular matrix is a square matrix where each row element is identical to the previous row but moves one position to the right [1]. One type of matrix is the RSLPFL (Row Skew Last-Plus-Last Left) circular matrix where the matrix with the first row is $(a_0,a_1,\dots,a_{n-1})$ denoted by *RSLPFLcircfr* $(a_0,a_1,\dots,a_{n-1})$, with the general form

$$A_n=\begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-3} & a_{n-2} & a_{n-1} \\ a_1 & a_2 & a_3 & \cdots & a_{n-2} & a_{n-1}+a_0 & -a_0 \\ a_2 & a_3 & a_4 & \cdots & a_{n-1}+a_0 & -a_0+a_1 & -a_1 \\ \vdots & \vdots & \vdots & ⋰ & \vdots & \vdots & \vdots \\ a_{n-3} & a_{n-2} & a_{n-1}+a_0 & \cdots & -a_{n-6}+a_{n-5} & -a_{n-5}+a_{n-4} & -a_{n-4} \\ a_{n-2} & a_{n-1}+a_0 & -a_0+a_1 & \cdots & -a_{n-5}+a_{n-4} & -a_{n-4}+a_{n-3} & -a_{n-3} \\ a_{n-1}+a_0 & -a_0+a_1 & -a_1 & \cdots & -a_{n-4}+a_{n-3} & -a_{n-3}+a_{n-2} & -a_{n-2} \end{bmatrix} \quad (1)$$

To get the entries in the $i+1$ row, the last entry is started by taking the first entry in the i row and multiplying by -1. After that, to determine the next entry, add up the first and last entries in the i row cyclically one position to the left[2]. Definitions and other matters related to the circular matrix are explained further in [3],[4],[5], and [6]. Several methods can be used to determine a

matrix inverse, that are matrix partitioning, decomposition methods, Gauss-Jordan elimination, and Adjoin methods. Other methods of determining the inverse of a matrix are also discussed in [7],[8], and [9]. One of the uses of the inverse of a matrix is solving a system of linear equation.
Several previous studies also discussed many matrices, including M.Eka Karnain et al. [10], Corry Corazon et al. [11], Rahmawati et al. [12], Zarifatul Aqila et al. [13], S.M Jauza et al. [14]. The writer is interested in doing this research to get the general form of the inverse of a particular form RSLPFLcircfr matrix using the adjoin method. With the uniqueness of the RSLPFLcircfr matrix, the authors suspect that the RSLPFLcircfr matrix has particular properties that it is expected to make it easier to solve the inverse of the matrix *RSLPFLcircfr*$\left(0,\frac{1}{b},0\right)$ with $a_{11}=0$, $a_{12}=\frac{1}{b}$, $a_{13}=0$ with $b\in R$ and $b\neq 0$ or can be written as follows

$$A_3=\begin{bmatrix} 0 & \frac{1}{b} & 0 \\ \frac{1}{b} & 0 & 0 \\ 0 & \frac{1}{b} & -\frac{1}{b} \end{bmatrix} \tag{2}$$

which will then determine the general form of the inverse of Equation (2).

Methods

This research is a literature study where there are steps to obtain the general inverse form of the *RSLPFLcircfr*$\left(0,\frac{1}{b},0\right)$ matrix with the power of positive integers, the following process will be given: First, given a matrix RSLPFL*circfr*$\left(0,\frac{1}{b},0\right)$ in Equation (2), the second is to determine the power of the matrix ${A_3}^2$ sampai ${A_3}^{10}$, The third is to predict the general form of the matrix ${A_3}^n$ with n is a positive integer, then prove the general form of the matrix ${A_3}^n$ using mathematical induction, determine $|{A_3}^2|$ to $|{A_3}^{10}|$, as well as predicting the general form of $|{A_3}^n|$ for n is a positive integer. Any conjecture in the general form obtained will be proved using the mathematical induction rules described in [15] and [16].

The following materials are provided to support the results and discussion.

Definisi 1 [17] If A is invertible, then the non-negative integer power of A is defined as:

$$A^0=I \text{ dan } A^n=A\,A\ldots A\,[\,n\ faktor]$$

And if A is invertible, then the negative integer power of A is defined as:

$$A^{-n}=(A^{-1})^n=A^{-1}A^{-1}\ldots A^{-1}\,[n\ faktor]$$

Teorema 1 [17] If A is an invertible and n is a non-negative integer, then:

a. A^{-1} is *invertible* dan $(A^{-1})^{-1}=A$

b. A^n is *invertible* dan $(A^n)^{-1}=A^{-n}=(A^{-1})^n$

c. kA is *invertible* for every non-zero scalar k and $(kA)^{-1}=k^{-1}A^{-1}$

Definisi 2[17] If A is an invertible matrix, then A^T is also invertible and $(A^T)^{-1}=(A^{-1})^T$.

Definisi 3 [17] If A is a square matrix, then the minor of the entry a_{ij} is denoted by M_{ij} and defined to be the determinant of the sub-matrix that remains after the i row and j column are removed from A. Numbers $(-1)^{i+j}M_{ij}$ are denoted by C_{ij} and are called cofactor of entry a_{ij}.

Teorema 2 [17] If we know a square matrix A of order $n \times n$, the determinant of matrix A can be calculated by multiplying the entries in a row or column by their cofactors and adding the resulting products, i.e. for each $1 \le i \le n$ and $1 \le j \le n$, then $\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \dots + a_{in}C_{in}$ (cofactor expansion along the i th row).

$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \dots + a_{nj}C_{nj}$ (cofactor expansion along the j-th column).

Definisi 4 [17] If A is a square matrix and if the matrix B is the same size with A such that $AB = BA = I$ then A is said to be a non-singular matrix and B is called the inverse of A. If the matrix B cannot be found, then A is said to be singular.

Teorema 3 [17] If A is an invertible matrix, then

$$A^{-1} = \frac{1}{|A|} adj(A)$$

Results and Discussion

This section contains the process for determining the general form of the inverse matrix RSLPFL*circfr* $\left(0, \frac{1}{b}, 0\right)$ to the power of positive integers. The first step is to determine the power of the matrix ${A_3}^2$ to ${A_3}^{10}$ as follows.

$${A_3}^2 = A_3 . A_3$$
$$= \begin{bmatrix} 0 & \frac{1}{b} & 0 \\ \frac{1}{b} & 0 & 0 \\ 0 & \frac{1}{b} & -\frac{1}{b} \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{b} & 0 \\ \frac{1}{b} & 0 & 0 \\ 0 & \frac{1}{b} & -\frac{1}{b} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1}{b^2} & 0 & 0 \\ 0 & \frac{1}{b^2} & 0 \\ \frac{1}{b^2} & -\frac{1}{b^2} & \frac{1}{b^2} \end{bmatrix}$$

In the same way, we get

$${A_3}^3 = \begin{bmatrix} 0 & \frac{1}{b^3} & 0 \\ \frac{1}{b^3} & 0 & 0 \\ -\frac{1}{b^3} & \frac{2}{b^3} & -\frac{1}{b^3} \end{bmatrix} \qquad {A_3}^7 = \begin{bmatrix} 0 & \frac{1}{b^7} & 0 \\ \frac{1}{b^7} & 0 & 0 \\ -\frac{3}{b^7} & \frac{4}{b^7} & -\frac{1}{b^7} \end{bmatrix}$$

$${A_3}^4 = \begin{bmatrix} \frac{1}{b^4} & 0 & 0 \\ 0 & \frac{1}{b^4} & 0 \\ \frac{2}{b^4} & -\frac{2}{b^4} & \frac{1}{b^4} \end{bmatrix} \qquad {A_3}^8 = \begin{bmatrix} \frac{1}{b^8} & 0 & 0 \\ 0 & \frac{1}{b^8} & 0 \\ \frac{4}{b^8} & -\frac{4}{b^8} & \frac{1}{b^8} \end{bmatrix}$$

$$A_3^5 = \begin{bmatrix} 0 & \frac{1}{b^5} & 0 \\ \frac{1}{b^5} & 0 & 0 \\ -\frac{2}{b^5} & \frac{3}{b^5} & -\frac{1}{b^5} \end{bmatrix} \qquad A_3^9 = \begin{bmatrix} 0 & \frac{1}{b^9} & 0 \\ \frac{1}{b^9} & 0 & 0 \\ -\frac{4}{b^9} & \frac{5}{b^9} & -\frac{1}{b^9} \end{bmatrix}$$

$$A_3^6 = \begin{bmatrix} \frac{1}{b^6} & 0 & 0 \\ 0 & \frac{1}{b^6} & 0 \\ \frac{3}{b^6} & -\frac{3}{b^6} & \frac{1}{b^6} \end{bmatrix} \qquad A_3^{10} = \begin{bmatrix} \frac{1}{b^{10}} & 0 & 0 \\ 0 & \frac{1}{b^{10}} & 0 \\ \frac{5}{b^{10}} & -\frac{5}{b^{10}} & \frac{1}{b^{10}} \end{bmatrix}$$

Furthermore, the general form of A_3^n is stated in Theorem 2 as follows:

Teorema 2. Given $A_3 = \begin{bmatrix} 0 & \frac{1}{b} & 0 \\ \frac{1}{b} & 0 & 0 \\ 0 & \frac{1}{b} & -\frac{1}{b} \end{bmatrix}$ with $b \in R$ and $b \neq 0$. Then

$$A_3^n = \begin{cases} \begin{bmatrix} 0 & \frac{1}{b^n} & 0 \\ \frac{1}{b^n} & 0 & 0 \\ -\frac{(n-1)}{2b^n} & \frac{n+1}{2b^n} & -\frac{1}{b^n} \end{bmatrix}, \text{ for odd } n. \\ \begin{bmatrix} \frac{1}{b^n} & 0 & 0 \\ 0 & \frac{1}{b^n} & 0 \\ \frac{n}{2b^n} & -\frac{n}{2b^n} & \frac{1}{b^n} \end{bmatrix}, \text{ for even } n. \end{cases} \qquad (3)$$

Bukti: Proof using mathematical induction as follows:

Suppose

$$p(n): A_3^n = \begin{bmatrix} 0 & \frac{1}{b^n} & 0 \\ \frac{1}{b^n} & 0 & 0 \\ -\frac{(n-1)}{2b^n} & \frac{n+1}{2b^n} & -\frac{1}{b^n} \end{bmatrix} \text{ for odd } n.$$

Proved as follows:

We will prove that for $n = 1$ then $p(1)$ is true.

$$p(1): A_3 = \begin{bmatrix} 0 & \frac{1}{b^n} & 0 \\ \frac{1}{b^n} & 0 & 0 \\ -\frac{(n-1)}{2b^n} & \frac{n+1}{2b^n} & -\frac{1}{b^n} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{1}{b} & 0 \\ \frac{1}{b} & 0 & 0 \\ 0 & \frac{1}{b} & -\frac{1}{b} \end{bmatrix}$$

By looking at Equation (2), then $p(1)$ is true.

Assume $n = k, p(k)$ is true, that is:

$$p(k): {A_3}^k = \begin{bmatrix} 0 & \frac{1}{b^k} & 0 \\ \frac{1}{b^k} & 0 & 0 \\ -\frac{(k-1)}{2b^k} & \frac{k+1}{2b^k} & -\frac{1}{b^k} \end{bmatrix}$$

Then it will be shown for $n = k+2$, $p(k+2)$ also be true, namely:

$$p(k+2): {A_3}^{k+2} = \begin{bmatrix} 0 & \frac{1}{b^{k+2}} & 0 \\ \frac{1}{b^{k+2}} & 0 & 0 \\ -\frac{(k+1)}{2b^{k+2}} & \frac{k+3}{2b^{k+2}} & -\frac{1}{b^{k+2}} \end{bmatrix}$$

The proof:

$${A_3}^{k+2} = {A_3}^k\ {A_3}^2$$

$$= \begin{bmatrix} 0 & \frac{1}{b^k} & 0 \\ \frac{1}{b^k} & 0 & 0 \\ -\frac{(k-1)}{2b^k} & \frac{k+1}{2b^k} & -\frac{1}{b^k} \end{bmatrix} \begin{bmatrix} \frac{1}{b^2} & 0 & 0 \\ 0 & \frac{1}{b^2} & 0 \\ \frac{1}{b^2} & -\frac{1}{b^2} & \frac{1}{b^2} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{1}{b^{k+2}} & 0 \\ \frac{1}{b^{k+2}} & 0 & 0 \\ -\frac{(k+1)}{2b^{k+2}} & \frac{k+3}{2b^{k+2}} & -\frac{1}{b^{k+2}} \end{bmatrix}$$

By looking at Equation (2), then $p(k+2)$ is true.

Steps (1) and (2) are correct for odd n.

$${A_3}^n = \begin{bmatrix} 0 & \frac{1}{b^n} & 0 \\ \frac{1}{b^n} & 0 & 0 \\ -\frac{(n-1)}{2b^n} & \frac{n+1}{2b^n} & -\frac{1}{b^n} \end{bmatrix}, \text{for odd } n$$

Based on the proof above, Theorem 2 is proven.
Next, we will prove that n is even.

$$p(n): {A_3}^n = \begin{bmatrix} \frac{1}{b^n} & 0 & 0 \\ 0 & \frac{1}{b^n} & 0 \\ \frac{n}{2b^n} & -\frac{n}{2b^n} & \frac{1}{b^n} \end{bmatrix}, \text{ for even } n.$$

1. We will prove that for $n = 2$ then $p(2)$ is true.
   $p(2): {A_3}^2 = A_3\, A_3$

$$= \begin{bmatrix} 0 & \frac{1}{b} & 0 \\ \frac{1}{b} & 0 & 0 \\ 0 & \frac{1}{b} & -\frac{1}{b} \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{b} & 0 \\ \frac{1}{b} & 0 & 0 \\ 0 & \frac{1}{b} & -\frac{1}{b} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{b^2} & 0 & 0 \\ 0 & \frac{1}{b^2} & 0 \\ \frac{1}{b^2} & -\frac{1}{b^2} & \frac{1}{b^2} \end{bmatrix}$$

   By looking at Equation (2), p(2) is true.
2. Assume $n = k, p(k)$ is true, that is:

$$p(k): {A_3}^k = \begin{bmatrix} \frac{1}{b^n} & 0 & 0 \\ 0 & \frac{1}{b^n} & 0 \\ \frac{n}{2b^n} & -\frac{n}{2b^n} & \frac{1}{b^n} \end{bmatrix}$$

   Then it will be shown to $n = k + 2$, $p(k+2)$ also to be true, namely:

$$p(k+2): {A_3}^{k+2} = \begin{bmatrix} \frac{1}{b^{k+2}} & 0 & 0 \\ 0 & \frac{1}{b^{k+2}} & 0 \\ \frac{k+2}{2b^{k+2}} & -\frac{k+2}{2b^{k+2}} & \frac{1}{b^{k+2}} \end{bmatrix}$$

   The proof:

$${A_3}^{k+2} = {A_3}^k\ {A_3}^2$$

$$= \begin{bmatrix} \frac{1}{b^k} & 0 & 0 \\ 0 & \frac{1}{b^k} & 0 \\ \frac{k}{2b^k} & -\frac{k}{2b^k} & \frac{1}{b^k} \end{bmatrix} \begin{bmatrix} \frac{1}{b^2} & 0 & 0 \\ 0 & \frac{1}{b^2} & 0 \\ \frac{2}{2b^2} & -\frac{2}{2b^2} & \frac{1}{b^2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{b^{k+2}} & 0 & 0 \\ 0 & \frac{1}{b^{k+2}} & 0 \\ \frac{k+2}{2b^{k+2}} & -\frac{k+2}{2b^{k+2}} & \frac{1}{b^{k+2}} \end{bmatrix}$$

By looking at Equation (2), then $p(k+2)$ is true.

Steps (1) and (2) are correct for even n.

$${A_3}^n = \begin{bmatrix} \frac{1}{b^n} & 0 & 0 \\ 0 & \frac{1}{b^n} & 0 \\ \frac{n}{2b^n} & -\frac{n}{2b^n} & \frac{1}{b^n} \end{bmatrix}, \text{ for even } n$$

Based on the proof above, Theorem 2 is proven.

The next step in this research is to determine the determinant of the matrix ${A_3}^2$ to ${A_3}^{10}$ by using the cofactor expansion method. The determinant matrix *RSLPFLcircfr*$\left(0, \frac{1}{b}, 0\right)$ of the order 3×3 to the power of positive integers is presented in Table 1 as follows:

Table 1. Determinant of Matrix *RSLPFLcircfr*$\left(0, \frac{1}{b}, 0\right)$ of the order 3×3

No	Matriks *RSLPFLcircfr*$\left(0, \frac{1}{b}, 0\right)$ ${A_3}^n$	$\lvert {A_3}^n \rvert$
1	${A_3}^2$	$\frac{1}{b^6}$
2	${A_3}^3$	$\frac{1}{b^9}$
3	${A_3}^4$	$\frac{1}{b^{12}}$
4	${A_3}^5$	$\frac{1}{b^{15}}$
5	${A_3}^6$	$\frac{1}{b^{18}}$
6	${A_3}^7$	$\frac{1}{b^{21}}$
7	${A_3}^8$	$\frac{1}{b^{24}}$
8	${A_3}^9$	$\frac{1}{b^{27}}$
9	${A_3}^{10}$	$\frac{1}{b^{30}}$

After getting the determinant of the matrix *RSLPFLcircfr*$\left(0,\frac{1}{b},0\right)$ of the order 3×3 to the power of positive integers, namely in Table 1, it can be estimated the general form of the determinant matrix *RSLPFLcircfr*$\left(0,\frac{1}{b},0\right)$ based on its recursive pattern, namely $|A_3{}^n| = \frac{1}{b^{3n}}$, for $n \in Z^+$. Based on these conjectures, we get the general form $|A_3{}^n|$ from Equation (2) which is stated in Theorem 3 as follows:

Theorem 3 Given $A_3 = \begin{bmatrix} 0 & \frac{1}{b} & 0 \\ \frac{1}{b} & 0 & 0 \\ 0 & \frac{1}{b} & -\frac{1}{b} \end{bmatrix}$ with $b \in R$ and $b \neq 0$. Then $|A_3{}^n| = \frac{1}{b^{3n}}$, for $n \in Z^+$.

Proof:

The general form of a power of a matrix $A_3{}^n$ is that there are two forms of matrices for n odd and n even. Based on Theorem 2, we get the general form $A_3{}^n$ (for n odd numbers) which will be used to prove the alleged general form of the power of the determinant using direct proof by cofactor expansion along the first row as follows:

$$A_3{}^n = \begin{bmatrix} 0 & \frac{1}{b^n} & 0 \\ \frac{1}{b^n} & 0 & 0 \\ -\frac{n-1}{2b^n} & \frac{n+1}{2b^n} & -\frac{1}{b^n} \end{bmatrix}, \text{ then}$$

$$|A_3{}^n| = 0\begin{vmatrix} 0 & 0 \\ \frac{n+1}{2b^n} & -\frac{1}{b^n} \end{vmatrix} - \frac{1}{b^n}\begin{vmatrix} \frac{1}{b^n} & 0 \\ -\frac{(n-1)}{2b^n} & -\frac{1}{b^n} \end{vmatrix} + 0\begin{vmatrix} \frac{1}{b^n} & 0 \\ -\frac{(n-1)}{2b^n} & \frac{n+1}{2b^n} \end{vmatrix}$$

$$= 0 - \frac{1}{b^n}\left(-\frac{1}{b^{2n}} - 0\right) + 0$$

$$= \frac{1}{b^{3n}}$$

So it is proved that $\left|A_3{}^n\right| = \frac{1}{b^{3n}}$, for odd n.

Furthermore, based on Theorem 2, the general form $A_3{}^n$ (for n even numbers) is obtained, which will be used to prove the alleged general form of the power of the determinant using direct proof with cofactor expansion along the first line as follows:

$$A_3{}^n = \begin{bmatrix} \frac{1}{b^n} & 0 & 0 \\ 0 & \frac{1}{b^n} & 0 \\ \frac{n}{2b^n} & -\frac{n}{2b^n} & \frac{1}{b^n} \end{bmatrix}, \text{ then}$$

$$|A_3{}^n| = \frac{1}{b^n}\begin{vmatrix} \frac{1}{b^n} & 0 \\ \frac{n}{2b^n} & \frac{1}{b^n} \end{vmatrix} - 0\begin{vmatrix} 0 & 0 \\ \frac{n}{2b^n} & \frac{1}{b^n} \end{vmatrix} + 0\begin{vmatrix} 0 & \frac{1}{b^n} \\ \frac{n}{2b^n} & -\frac{n}{2b^n} \end{vmatrix}$$

$$= \frac{1}{b^n}\left(-\frac{1}{b^{2n}} - 0\right) - 0 + 0$$

$$= \frac{1}{b^{3n}}$$

So it is proved that $\left|A_3{}^n\right| = \frac{1}{b^{3n}}$, for even n.

Based on the proof above, then Theorem 3 is proven.

The next step is to get the general form of the cofactor matrix from the power of the matrix *RSLPFLcircfr*$\left(0, \frac{1}{b}, 0\right)$ based on Equation (4) which is stated in Theorem 4 as follows:

Theorem 4 Given the matrix RSLPFL*circfr* $\left(0, \frac{1}{b}, 0\right)$ is $A_3 = \begin{bmatrix} 0 & \frac{1}{b} & 0 \\ \frac{1}{b} & 0 & 0 \\ 0 & \frac{1}{b} & -\frac{1}{b} \end{bmatrix}$, with $b \in R$ and $b \neq 0$.

Then obtained:

$$C_3{}^n = \begin{cases} \begin{bmatrix} 0 & \frac{1}{b^{2n}} & \frac{n+1}{2b^{2n}} \\ \frac{1}{b^{2n}} & 0 & -\frac{(n-1)}{2b^{2n}} \\ 0 & 0 & -\frac{1}{b^{2n}} \end{bmatrix}, \text{ for odd } n \\ \\ \begin{bmatrix} \frac{1}{b^{2n}} & 0 & -\frac{n}{2b^{2n}} \\ 0 & \frac{1}{b^{2n}} & \frac{n}{2b^{2n}} \\ 0 & 0 & \frac{1}{b^{2n}} \end{bmatrix}, \text{ for even } n \end{cases} \tag{4}$$

Proof:

We will prove that $C_3{}^n = \begin{bmatrix} 0 & \frac{1}{b^{2n}} & \frac{n+1}{2b^{2n}} \\ \frac{1}{b^{2n}} & 0 & -\frac{(n-1)}{2b^{2n}} \\ 0 & 0 & -\frac{1}{b^{2n}} \end{bmatrix}$, for odd n.

Based on Theorem 2, we get the general form $A_3{}^n$ (for n odd numbers) which will be used to prove the alleged general form of the power of the cofactor matrix using direct proof as follows:

$$A_3{}^n = \begin{bmatrix} 0 & \frac{1}{b^n} & 0 \\ \frac{1}{b^n} & 0 & 0 \\ -\frac{(n-1)}{2b^n} & \frac{n+1}{2b^n} & -\frac{1}{b^n} \end{bmatrix}$$

$$C_{11} = (-1)^2 \begin{vmatrix} \frac{1}{b^n} & 0 \\ -\frac{n}{2b^n} & \frac{1}{b^n} \end{vmatrix} = \frac{1}{b^{2n}}$$

$$C_{12} = (-1)^3 \begin{vmatrix} 0 & 0 \\ \frac{n}{2b^n} & \frac{1}{b^n} \end{vmatrix} = 0$$

$$C_{13} = (-1)^4 \begin{vmatrix} 0 & \frac{1}{b^n} \\ \frac{n}{2b^n} & -\frac{n}{2b^n} \end{vmatrix} = -\frac{n}{2b^{2n}}$$

$$C_{21} = (-1)^3 \begin{vmatrix} 0 & 0 \\ \frac{n}{2b^n} & \frac{1}{b^n} \end{vmatrix} = 0$$

$$C_{22} = (-1)^4 \begin{vmatrix} \frac{1}{b^n} & 0 \\ \frac{n}{2b^n} & \frac{1}{b^n} \end{vmatrix} = \frac{1}{b^{2n}}$$

$$C_{23} = (-1)^5 \begin{vmatrix} \frac{1}{b^n} & 0 \\ \frac{n}{2b^n} & -\frac{n}{2b^n} \end{vmatrix} = \frac{n}{2b^{2n}}$$

$$C_{31} = (-1)^4 \begin{vmatrix} 0 & 0 \\ \frac{1}{b^n} & 0 \end{vmatrix} = 0$$

$$C_{32} = (-1)^5 \begin{vmatrix} \frac{1}{b^n} & 0 \\ 0 & 0 \end{vmatrix} = 0$$

$$C_{33} = (-1)^6 \begin{vmatrix} \frac{1}{b^n} & 0 \\ 0 & \frac{1}{b^n} \end{vmatrix} = \frac{1}{b^{2n}}$$

From the above calculations, obtained:

$$C_3{}^n = \begin{bmatrix} 0 & \frac{1}{b^{2n}} & \frac{n+1}{2b^{2n}} \\ \frac{1}{b^{2n}} & 0 & -\frac{(n-1)}{2b^{2n}} \\ 0 & 0 & -\frac{1}{b^{2n}} \end{bmatrix}$$

So that $C_3{}^n$ for n odd numbers is proved.

So that $C_3{}^n = \begin{bmatrix} \frac{1}{b^{2n}} & 0 & -\frac{n}{2b^{2n}} \\ 0 & \frac{1}{b^{2n}} & \frac{n}{2b^{2n}} \\ 0 & 0 & \frac{1}{b^{2n}} \end{bmatrix}$, for even n

Based on Theorem 2, we get the general form of the power of $A_3{}^n$ (for n even numbers) which will be used to prove the conjecture of the general form of the power of the cofactor matrix using direct proof as follows:

$$A_3{}^n = \begin{bmatrix} 0 & \frac{1}{b^n} & 0 \\ \frac{1}{b^n} & 0 & 0 \\ -\frac{(n-1)}{2b^n} & \frac{n+1}{2b^n} & -\frac{1}{b^n} \end{bmatrix}$$

$$C_{11} = (-1)^2 \begin{vmatrix} 0 & 0 \\ \frac{n+1}{2b^n} & -\frac{1}{b^n} \end{vmatrix} = 0$$

$$C_{12} = (-1)^3 \begin{vmatrix} \frac{1}{b^n} & 0 \\ \frac{-(n-1)}{2b^n} & -\frac{1}{b^n} \end{vmatrix} = \frac{1}{b^{2n}}$$

$$C_{13} = (-1)^4 \begin{vmatrix} \frac{1}{b^n} & 0 \\ \frac{-(n-1)}{2b^n} & \frac{n+1}{2b^n} \end{vmatrix} = \frac{n+1}{2b^{2n}}$$

$$C_{21} = (-1)^3 \begin{vmatrix} \frac{1}{b^n} & 0 \\ \frac{n+1}{2b^n} & -\frac{1}{b^n} \end{vmatrix} = \frac{1}{b^{2n}}$$

$$C_{22} = (-1)^4 \begin{vmatrix} 0 & 0 \\ \frac{-(n-1)}{2b^n} & -\frac{1}{b^n} \end{vmatrix} = 0$$

$$C_{23} = (-1)^5 \begin{vmatrix} 0 & \frac{1}{b^n} \\ \frac{-(n-1)}{2b^n} & \frac{n+1}{2b^n} \end{vmatrix} = -\frac{(n-1)}{2b^{2n}}$$

$$C_{31} = (-1)^4 \begin{vmatrix} \frac{1}{b^n} & 0 \\ 0 & 0 \end{vmatrix} = 0$$

$$C_{32} = (-1)^5 \begin{vmatrix} 0 & 0 \\ \frac{1}{b^n} & 0 \end{vmatrix} = 0$$

$$C_{33} = (-1)^6 \begin{vmatrix} 0 & \frac{1}{b^n} \\ \frac{1}{b^n} & 0 \end{vmatrix} = -\frac{1}{b^{2n}}$$

From the calculation above, it is obtained:

$$C_3{}^n = \begin{bmatrix} \frac{1}{b^{2n}} & 0 & -\frac{n}{2b^{2n}} \\ 0 & \frac{1}{b^{2n}} & \frac{n}{2b^{2n}} \\ 0 & 0 & \frac{1}{b^{2n}} \end{bmatrix}$$

So that $C_3{}^n$ for n even numbers are proved.

Based on the proof above, Theorem 4 is proven.

Furthermore, the general form of the inverse power of the matrix *RSLPFLcircfr*$\left(0, \frac{1}{b}, 0\right)$ will be estimated based on Equation (3) and Equation (4) which are stated in Theorem 4 as follows:

Theorem 5 Given the matrix RSLPFL*circfr* $\left(0, \frac{1}{b}, 0\right)$ in Equation (2), namely :

$A_3 = \begin{bmatrix} 0 & \frac{1}{b} & 0 \\ \frac{1}{b} & 0 & 0 \\ 0 & \frac{1}{b} & -\frac{1}{b} \end{bmatrix}$, with $b \in R$ and $b \neq 0.$ Then we get:

$$(A_3{}^n)^{-1} = \begin{cases} \begin{bmatrix} 0 & b^n & 0 \\ b^n & 0 & 0 \\ \frac{1}{2}(n+1)b^n & -\frac{1}{2}(n+1)b^n & b^n \end{bmatrix}, \text{ for odd } n \\ \begin{bmatrix} b^n & 0 & 0 \\ 0 & b^n & 0 \\ -\frac{1}{2}nb^n & \frac{1}{2}nb^n & b^n \end{bmatrix}, \text{ for even } n \end{cases} \quad (5)$$

Bukti: The proof of the theorem above is proven using the adjoin method. Based on Equation (3) and Equation (4) for odd n, it is obtained using Theorem 1 as follows:

$$(A_3{}^n)^{-1} = \frac{1}{|A_3{}^n|}[adj\ (A_3{}^n)]$$

$$= \frac{1}{|A_3{}^n|}[C_3{}^n]^T$$

$$= b^{3n}\begin{bmatrix} 0 & \frac{1}{b^{2n}} & \frac{n+1}{2b^{2n}} \\ \frac{1}{b^{2n}} & 0 & -\frac{(n-1)}{2b^{2n}} \\ 0 & 0 & \frac{1}{b^{2n}} \end{bmatrix}^T$$

$$= b^{3n}\begin{bmatrix} 0 & \frac{1}{b^{2n}} & 0 \\ \frac{1}{b^{2n}} & 0 & 0 \\ \frac{n+1}{2b^{2n}} & -\frac{(n-1)}{2b^{2n}} & \frac{1}{b^{2n}} \end{bmatrix}$$

$$(A_3{}^n)^{-1} = \begin{bmatrix} 0 & b^n & 0 \\ b^n & 0 & 0 \\ \frac{1}{2}(n+1)b^n & -\frac{1}{2}(n+1)b^n & b^n \end{bmatrix}$$

Based on Equation (3) and Equation (4) for even n, it is obtained using Theorem 1 as follows:

$$(A_3{}^n)^{-1} = \frac{1}{|A_3{}^n|}[adj\ (A_3{}^n)]$$

$$= \frac{1}{|A_3{}^n|}[C_3{}^n]^T$$

$$= b^{3n}\begin{bmatrix} \frac{1}{b^{2n}} & 0 & -\frac{n}{2b^{2n}} \\ 0 & \frac{1}{2b^{n}} & \frac{n}{2b^{2n}} \\ 0 & 0 & \frac{1}{b^{2n}} \end{bmatrix}^T$$

$$= b^{3n}\begin{bmatrix} \frac{1}{b^{2n}} & 0 & 0 \\ 0 & \frac{1}{b^{2n}} & 0 \\ -\frac{n}{2b^{2n}} & \frac{n}{2b^{2n}} & \frac{1}{b^{2n}} \end{bmatrix}$$

$$(A_3{}^n)^{-1} = \begin{bmatrix} b^n & 0 & 0 \\ 0 & b^n & 0 \\ -\frac{1}{2}nb^n & \frac{1}{2}nb^n & b^n \end{bmatrix}$$

Conclusion

In this paper, a conclusion is obtained for the matrix *RSLPFLcircfr* $\left(0,\frac{1}{b},0\right)$ of order 3×3 to the power of positive integers using an adjoin in Equation (5), then also obtained the general form of the matrix determinant *RSLPFLcircfr* $\left(0,\frac{1}{b},0\right)$. This results in the determinant's general form in Equation (6). The general form of the cofactor matrix in Equation (7) is also obtained and the general form of the inverse matrix *RSLPFLcircfr* $\left(0,\frac{1}{b},0\right)$ of the order 3×3 to the power of a positive integer.

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