Vertex Labeled Energy of Edge-Removed Complete Graphs

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Abstract

Suppose Γ is vertex labeled graph by its degree. The (i,j)-th entry of vertex labeled matrix of Γ is label sum of different vertices v_i and v_j if there are paths between them, and 0 otherwise. Vertex labeled energy of Γ is absolute sum of its vertex labeled matrix eigenvalues. In this paper, we provide value of vertex labeled energy of edge-removed complete graph

Keywords: vertex labeled graph, vertex labeled matrix, vertex labeled energy, edge-removed complete graph, eigenvalues

Introduction

Graph energy is a branch of spectral graph theory. First introduced by Gutman in 1978, energy of graph is defined by sum of the absolute values of its adjacency matrix's eigenvalues [1]. There are many various expansions in graph energy definition depending on the type of matrix used, such as Laplacian energy[2], distance energy[3], Randic energy[4], Sombor energy[5], and so on. Inspired by binary labeled energy[6], Kavita Permi, Y. R. L. Indrani, K. N. Prakasha introduced definition of vertex labeled energy, and determined the vertex labeled energy of some special graphs such as complete graphs, star graphs, wheel graphs, cocktail party graphs, crown graphs, complete bipartite graphs, double star graphs, friendship graphs, and the complement of double star graphs [7].

Complete graph, denoted by K_n, is a graph where every pair of distinct vertices is connected by single edge. The edge-removed complete graph, denoted by $K_n - e$, is a graph formed by remove an arbitrary edge of complete graph [8]. This paper provides vertex labeled energy of edge-removed complete graph and its association to vertex labeled energy of complete graph

Basic Concepts of Graph Theory and Linear Algebra

Definition 1 [9]**.** A graph Γ is a pair of sets (V, X). V is the non-empty set of vertices and X is the set of edges connecting a pair of vertices.

Definition 2[10]**.** A complete graph is a simple graph in which every two distinct vertices are adjacent. A complete graph is denoted by $K_n, n \geq 1$ and n natural numbers.

Definition 3 [11]**.** A labeling of a graph is a mapping of a set of graph elements to a set of positive integers. Vertex labeling is a label whose domain is the set of vertices. If the domain is a set of edges, it is called an edge labeling. And if the domain is vertex and side then it is called total labeling.

Definition 4[9]**.** If A is a matrix $n \times n$ then the non-zero vector x in R^n is called an eigenvector of A if there is a scalar λ such as $Ax = \lambda x$.

Definition 5 [9]**.** A diagonal matrix is a square matrix in which all the elements outside the main diagonal are zero.

Definition 6[12]**.** A block matrix is a matrix that is partitioned into smaller matrices, or blocks, arranged in a rectangular or square configuration, with each block representing a separate sub-matrix. These blocks can be of any size, and their arrangement allows for more efficient representation and manipulation of structured data in various mathematical and computational applications.

Definition 7[11]**.** The characteristic polynomial of a square matrix is a polynomial equation whose roots are the eigenvalues of that matrix. Let A is square matrix, then characteristic polynomial of A is

$$C(\lambda) = det(A - I\lambda) = det(\lambda I - A) \quad (1)$$

with λ the eigenvalues of A

Definition 8[11]**.** The spectrum of a graph, also known as the graph spectrum, refers to the set of eigenvalues of the graph's adjacency matrix or its Laplacian matrix. Let λ_i is the eigenvalue of Γ with multiplicity m_i. Suppose $\lambda_1 > \lambda_2 > \cdots > \lambda_k$, graph spectrum of Γ is So

$$spec(\Gamma) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ m_1 & m_2 & \cdots & m_k \end{pmatrix} \quad (2)$$

Definition 9. Energy of matrix is sum of the absolute of its eigenvalues.

Definition 10 [7]. Let L is vertex labelling of Γ is defined by $L(v) = \deg(v)$ for vertex v in graph Γ. Vertex labeled matrix of Γ, denoted by $V_L(\Gamma)$, is matrix $[m_{ij}]$ with

$$m_{ij} = \begin{cases} L(v_i) + L(v_j), & if\ there\ are\ paths\ between\ different\ vertices\ v_i\ and\ v_j \\ 0 \quad , & otherwise \end{cases}$$

Definition 11 [7]**.** The vertex labeled energy of graph Γ is energy of vertex labeled matrix of Γ.

Theorem 12 [7]. The vertex labeled energy of complete graph with n vertices, $n \geq 2$, is $4n^2 - 8n + 4$.

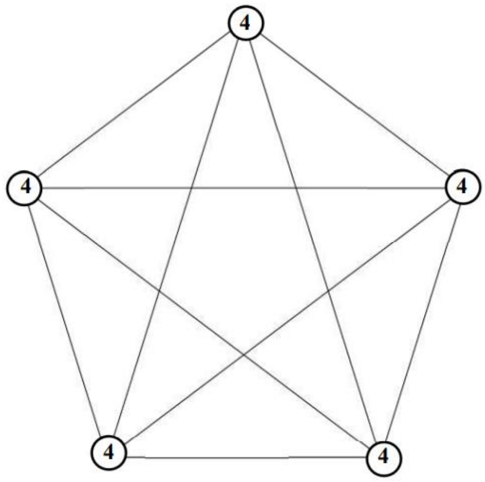

Figure 1. Vertex labeled K_5 graph

For example, the non-diagonal entries of vertex labeled of K_5 is 8 and 0 for all diagonal entries. Therefore, we get spectrum of vertex labeled matrix of K_5 is

$$Spec(V_l(K_5)) = \begin{pmatrix} 32 & -8 \\ 1 & 4 \end{pmatrix},$$

and the vertex labeled energy of K_5 is 64.

Results and Discussion

Lemma 13[13]–[15]**.** Let A be a block diagonal matrix of order n and square matrix A_i is diagonal entries of A, then the determinant of A satisfies:

$$\det(\boldsymbol{A}) = \prod_{i \geq 0} \det(\boldsymbol{A_i}) \tag{3}$$

Definition 14. A edge-removed complete graph, denoted by $K_n - e$, is a graph complete graph K_n with one edge removed.

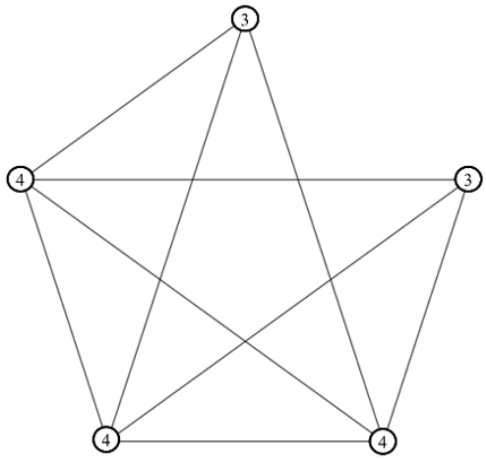

Figure 2. Vertex labeled $K_5 - e$ graph

Theorem 15. For integer $n \geq 3$, vertex labeled energy of edge-removed complete graph with n vertices is $4n^2 - 12n + 4 + 2\sqrt{n^4 - 2n^3 - 5n^2 + 16n - 11}$

Proof. The vertex labeled matrix of $K_n - e$ is given by

$$V_L(K_n - e) = \begin{bmatrix} 0 & 2n-4 & 2n-3 & 2n-3 & & 2n-3 & 2n-3 \\ 2n-4 & 0 & 2n-3 & 2n-3 & \cdots & 2n-3 & 2n-3 \\ 2n-3 & 2n-3 & 0 & 2n-2 & & 2n-2 & 2n-2 \\ 2n-3 & 2n-3 & 2n-2 & 0 & & 2n-2 & 2n-2 \\ & & \vdots & & \ddots & & \vdots \\ 2n-3 & 2n-3 & 2n-2 & 2n-2 & \cdots & 0 & 2n-2 \\ 2n-3 & 2n-3 & 2n-2 & 2n-2 & & 2n-2 & 0 \end{bmatrix}$$

Let $C_L(\lambda)$ is the characteristic polynomial of $V_L(K_n - e)$ i.e., $C_L(\lambda) = \det\big(\lambda I - V_L(K_n - e)\big)$. By elementary row operations, we obtained $C_L(\lambda)$ is equal to

$$det \begin{bmatrix} \lambda + (2n-4) & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda - (2n-4) & -(2n-3)(n-2) & 0 & 0 & \cdots & 0 & 0 \\ 0 & -2(2n-3) & \lambda - (2n-2)(n-3) & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \lambda + (2n-2) & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda + (2n-2) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & \lambda + (2n-2) & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \lambda + (2n-2) \end{bmatrix}$$

We can write

$$C_L(\lambda) = det\begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{bmatrix}$$

with

$$A_1 = [\lambda + (2n-4)],$$

$$A_2 = \begin{bmatrix} \lambda - (2n-4) & -(2n-3)(n-2) \\ -2(2n-3) & \lambda - (2n-2)(n-3) \end{bmatrix},$$

and

$$A_3 = \begin{bmatrix} \lambda + (2n-2) & 0 & \cdots & 0 \\ 0 & \lambda + (2n-2) & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \lambda + (2n-2) \end{bmatrix}.$$

By using lemma 13,

$$C_L(\lambda) = det(A_1) \cdot det(A_2) \cdot det(A_3)$$

Therefore, we obtained

$$C_L(\lambda) = \big(\lambda + (2n-4)\big)(\lambda^2 - (2n^2 - 6n + 2)\lambda - (4n^3 - 24n^2 + 44n - 24) - 2(n - 2)(2n-3)^2)\big(\lambda + (2n-2)\big)^{n-3}.$$

The spectrum of vertex labeled matrix of $K_n - e$ is

$$spec(V_L(K_n - e)) = \begin{pmatrix} n^2 - 3n + 1 + \sqrt{n^4 - 2n^3 - 5n^2 + 16n - 11} & 1 \\ -(2n-4) & 1 \\ n^2 - 3n + 1 - \sqrt{n^4 - 2n^3 - 5n^2 + 16n - 11} & 1 \\ -(2n-2) & n-3 \end{pmatrix}^T$$

Then, vertex labeled energy of $K_n - e$ is

$$4n^2 - 12n + 4 + 2\sqrt{n^4 - 2n^3 - 5n^2 + 16n - 11}. \blacksquare$$

For example, we will provide value of vertex labeled energy of $K_5 - e$. Consider the vertex labeled matrix of $K_5 - e$,

$$V_L(K_5 - e) = \begin{bmatrix} 0 & 6 & 7 & 7 & 7 \\ 6 & 0 & 7 & 7 & 7 \\ 7 & 7 & 0 & 8 & 8 \\ 7 & 7 & 8 & 0 & 8 \\ 7 & 7 & 8 & 8 & 0 \end{bmatrix}$$

We can obtain characteristics polynomial of $V_L(K_5 - e))$

$$C_L(\lambda) = (\lambda + 6)(\lambda^2 - 22\lambda - 198)(\lambda + 8)^2$$

Therefore, the spectrum of $V_L E(K_5 - e)$ is

$$spec(V_L(K_5 - e)) = \begin{pmatrix} 11 + \sqrt{319} & -6 & 11 - \sqrt{319} & -8 \\ 1 & 1 & 1 & 2 \end{pmatrix}$$

And the vertex labeled energy graph $of\ K_5 - e$ is $44 + 2\sqrt{319}$.

Corollary 15. For integer $n \geq 3$, the vertex labeled energy of $K_n - e$ is less than vertex labelled energy of K_n.

Proof. We will prove that $E(V_L(K_n)) - E(V_L(K_n - e)) > 0$

We have

$$E(V_L(K_n)) - E(V_L(K_n - e))$$

$$= (4n^2 - 8n + 4) - \left(4n^2 - 12n + 4 + 2\sqrt{n^4 - 2n^3 - 5n^2 + 16n - 11}\right)$$

$$= 4n - 2\sqrt{n^4 - 2n^3 - 5n^2 + 16n - 11}.$$

Let $f(n) = 4n - 2\sqrt{n^4 - 2n^3 - 5n^2 + 16n - 11}$

For $n = 3$, we get $f(3) = 12 - 2\sqrt{19} > 0$

Assume for $n = k$ that $f(k) = 4k - 2\sqrt{k^4 - 2k^3 - 5k^2 + 16k - 11} > 0$

And for $n = k + 1$ than

$$f(k+1) = 4(k+1) - 2\sqrt{(k+1)^4 - 2(k+1)^3 - 5(k+1)^2 + 16(k+1) - 11}$$
$$= 4(k+1) - 2\sqrt{k^4 - 2k^3 - 5k^2 + 4k - 1}$$
$$> 4k - 2\sqrt{k^4 - 2k^3 - 5k^2 + 4k - 1 + 12k - 11 - (12k - 11)}$$
$$> 4k - 2\sqrt{k^4 - 2k^3 - 5k^2 + 16k - 11} = f(k) > 0.$$

In conclusion, we get that $4n - 2\sqrt{n^4 - 2n^3 - 5n^2 + 16n - 11} > 0$ for integer $n \geq 3$ ■

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