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Dynamical Analysis of Atlantic Cod Population Described in A Leslie-Gower Model with Allee Effect and Non-Linear Harvesting

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Abstract

The fishing industry has fished the Atlantic cod population for a millennium. As fishing became greater, the Atlantic cod population declined. During this crisis, fishing declined drastically, but the species is still struggling to bounce back. This is due to the growing number of sharks and other North Atlantic invertebrates. The effects of predation from predators cause a decrease in the per capita fertility rate below the minimum density or can be called the Allee effect. In addition, human intervention such as harvesting can disrupt the predator-prey system. Based on this phenomenon, a Leslie-Gower model involving additive Allee effects and non-linear harvesting is considered to describe population dynamics. Further research is done on the existence and stability of equilibrium points as well as the positivity, permanence, and boundedness of solutions. It was found that the permanence requirement was used as a condition to ensure the two populations lived together. Numerical simulations are performed in the last section to bolster the analysis's findings.

Keywords: Allee effect, Leslie-Gower model, non-linear harvesting
MSC2020: 92B05, 92D25, 93C15.

Abstrak

Populasi ikan Atlantik cod telah ditangkap selama seribu tahun. Seiring penangkapan ikan yang semakin besar, populasi ikan Atlantik cod semakin menurun. Selama krisis ini, penangkapan ikan berkurang secara drastis, namun spesies ini masih kesulitan untuk bangkit kembali. Hal ini disebabkan karena pertumbuhan jumlah ikan hiu dan invertebrata Atlantik Utara lainnya. Akibat pemangsaan dari predator menyebabkan penurunan tingkat kesuburan per kapita dibawah kepadatan minimum atau dapat disebut dengan efek Allee. Selain itu, intervensi manusia seperti pemanenan dapat mengganggu sistem predator-prey. Berdasarkan fenomena ini, dipertimbangkan model Leslie-Gower yang melibatkan additive Allee effect dan pemanenan non-linear untuk menggambarkan dinamika populasi yang terjadi. Kepositifan, keabadian, dan keterbatasan solusi, eksistensi dan kestabilan titik equilibrium dikaji lebih lanjut. Ditemukan bahwa syarat keabadian menjadi syarat untuk menjamin

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kedua populasi hidup bersama. Pada bagian akhir, dilakukan simulasi numerik untuk mendukung hasil teoritis yang diperoleh.

Kata Kunci: Efek Allee, model Leslie-Gower, pemanenan non-linear
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Introduction

Reportedly from articles such as [1], Atlantic cod and closely related species are unique among fish in that they have two separate anal fins found along the ventral surface and three distinct dorsal fins found along the back. The majority of the life of Atlantic cod is spent on or close to the sea floor, despite the fact that they are sometimes spotted feeding high above the water. They consume a variety of food sources, such as invertebrates, American lobsters, and vertebrate fish species. Adult Atlantic cod are devoured by giant sharks, while young are eaten by a range of medium-sized predators, including cannibalistic adults.

Atlantic cod has been caught for a thousand years. The first people to catch these fish were Northern European fishermen. The Atlantic cod has improved fisheries and supported the coastal economies of North America for generations. During the initial stages of Caribbean colonization, dried and salted Atlantic fish gained significance as a food source. Unfortunately, as fishing became larger, the Atlantic cod population declined, and in the late 1990s, the fisheries sector collapsed. During this crisis, fishing was drastically reduced, but the species is still struggling to bounce back. This is due to the growth in the number of sharks and other North Atlantic invertebrates. Experts concur that the demise of the Atlantic cod has had a profound effect on the North Atlantic food web, and the species is currently regarded as vulnerable to extinction.

Based on the empirical data above, it is known that interactions between Atlantic cod and shark populations occur in an ecosystem. The interaction of these two populations can be modeled into a mathematical model or called a predator-prey model. So far, there are 2 predator-prey models that are popularly used in researching population dynamics, namely the Leslie-Gower model and the Lotka-Voltera model. In the Lotka-Voltera model, the growth rate of the predator depends on the consumption rate of the predator [2], [3]. In the Leslie-Gower model, on the other hand, the rate of increase of predators is determined by their carrying capacity, which is directly related to the quantity of prey [4].

An interesting phenomenon was found in the data above, namely the growth of the Atlantic cod fish which is threatened with extinction is caused by the growth of sharks and North Atlantic invertebrates as predators. Reduced prey density due to predation from predators is a common mechanism that effects survival rates. We can refer to this phenomena as the Allee effect. The Allee effect causes per capita fertility rates to fall below the minimum density [5]. More specifically, population density and fertility rate are positively correlated according to the Allee effect [6]. A large Allee effect in this case suggests that there is a threshold population level [7], whereas a small Allee effect suggests the contrary [8]. The Allee effect occurs because of several other phenomena, such as low population due to internal competition and difficulty in finding mates [9].

The Allee effect mechanism has been simulated in the form of a mathematical function. Allee effects can be strong or weak, but usually ecologists consider Allee effects to be strong. The double Allee effect equation expresses the strongest Allee effect that occurs most frequently (see [10], [11]) because this function shows two or more Allee effects that occur in the same population. Meanwhile, one mechanism for the Allee effect that occurs is expressed in the multiplicative Allee effect equation (see [12]) or vice versa, stated as the additive Allee effect (see [9], [13], [14]).

Human activities such as harvesting have the potential to disrupt the predator-prey balance in natural ecosystems. Therefore, when we examine ecological dynamics, it is important to consider the effects of harvesting. In order to observe the maximum sustainable and optimal total returns of tax policy, Ghosh and Tar [15] take into account alternative forms of the harvest function. Maintaining environmental balance and fostering system stability are two major goals of the harvesting function. Generally, the functions used in ecological systems are proportional permanent functions [16], [17], [18].

In both theoretical and practical research, the interaction of Allee effects and harvest effects on predator-prey systems has drawn interest. Wu et al. [19] examined powerful Allee effects and threshold harvesting in the context of a diffuse predator-prey system's complex dynamics problem. They discovered that adding threshold harvesting and significant Allee effects to the system can make the spatiotemporal system more complex. Shang and Qiao [20] looked into the double Allee effect, nonlinear harvesting, and a modified Leslie-Gower model with a Holling type IV response function. They discovered that there were many instability and bifurcations in the system. A predator-prey model with high Allee effects and impulsive harvesting was examined by Wei and Chen [21]. They discovered that heteroclinic bifurcation and order-1 periodic solutions are admissible in this system.

Harvest factors in predator-prey systems are very important to study today. Without considering harvest effort, ecosystem dynamics cannot be accurately represented. The populations of predators and prey may be directly effected by harvesting. It can also aid in the improvement and verification of ecological models. Thus, it is imperative to have understanding of how harvesting affects ecological dynamics, particularly in light of the Allee effect. In addition, the non-linear harvest function is considered a more realistic function to analyze than the proportional harvest function, making it valuable for understanding ecological dynamics with the Allee effect. Thus, this research concentrates on a detailed analysis of non-linear harvest functions by taking the Allee effect, particularly the additive Allee effect, into account, based on prior empirical evidence.

Mathematical Formulation

This section develops the predator-prey mathematical model based on the Leslie-Gower model [22]. The Leslie-Gower model have applied in [4], [9], [23] as follows.

$$\begin{aligned} \frac{dx}{dt} &= rx\left(1-\frac{x}{K}\right)-pxy, \\ \frac{dy}{dt} &= sy\left(1-\frac{y}{nx}\right), \end{aligned} \tag{1}$$

where $x(0) > 0$ and $y(0) > 0$. The number of prey and predator at time t with $x(t) > 0$ and $y(t) \geq 0$ for $t \geq 0$ is written by $x(t)$ and $y(t)$, respectively. Furthermore, $r, s, K, p,$ and n are explained in [9], [23], namely the intrinsic growth rate of prey, intrinsic growth rate of predator, carrying capacity of prey in the absence of predator, consumption rate of predator, and a measure of food quality that prey provides for conversion into predator birth, respectively.

Humans are essential to the existence of predators and prey in an ecosystem. For example, it can see from the predator-prey interaction between sharks and Atlantic cod. In fact, human often hunt sharks because they have economic value. Meanwhile, sharks only eat adult Atlantic cod. On the other hand, Atlantic cod is affected by Allee's effect due to continuous weather changes [1].

To describe predator-prey interaction as above, we consider several assumptions, namely 1) we consider that the interaction between Atlantic cod as prey and shark as predator follows Leslie-Gower model, 2) Atlantic cod became extinct due to the Allee effect which is described by the additive Allee effect function as in [13], [14], [16], and 3) we also assumed that only shark have important economic value and nonlinear harvesting rates are expected to reflect this. Thus, the following is a possible rewrite of system (1).

$$\begin{aligned}\frac{dx}{dt} &= rx\left(1-\frac{x}{K}-\frac{m}{x+b}\right)-pxy,\\ \frac{dy}{dt} &= sy\left(1-\frac{y}{nx}\right)-\frac{qEy}{m_1E+m_2y},\end{aligned} \tag{2}$$

where $x(0) > 0$ and $y(0) > 0$. In system (2), we can find important parts and parameters. The term $\frac{mx}{x+b}$ is the Allee function which shows that prey density is reduced, where the Allee parameters b and m follow biological conditions, that is 1) $0 < m < b$ for the small Allee and 2) $m > b$ for the high Allee [9]. Meanwhile, E and q positive constants represent individual harvest effort and fishing power coefficient, respectively. Parameters m_1 and m_2 are appropriate positive constants [23].

From system (2), we can simplify the system by scaling the parameters. The scaling carried out is the scaling of the independent variables time (t), and the attachment variables, namely prey (x) and predator (y). The scaling introduced is $(x, y, t) \to \left(Kx, \frac{ry}{p}, \frac{t}{r}\right)$. The scaling for system (2) produces a reduced system as follows.

$$\begin{aligned}\frac{dx}{dt} &= x\left(1-x-\frac{\alpha}{x+\beta}\right)-xy,\\ \frac{dy}{dt} &= \rho y\left(1-\frac{\gamma y}{x}\right)-\frac{hy}{c+y},\end{aligned} \tag{3}$$

with

$$x(0) > 0, y(0) > 0, \tag{4}$$

where

$$\alpha = \frac{m}{rK}, \beta = \frac{b}{K}, \rho = \frac{s}{r}, \gamma = \frac{r}{npK}, h = \frac{qpE}{m_2r^2}, c = \frac{m_1pE}{m_2r}.$$

Here, the system (3) has several conditions, that is 1) the Allee's small effect if $0 < \alpha < \beta$ and 2) the Allee's high effect if $\alpha > \beta$. Based on biological condition, the solution of system (3) always lies in the first quadrant defined by $R_+^2 = \{(x, y) \in \mathbb{R}^2 | x \geq 0, y \geq 0\}$.

From the system simplification above, we can reduce several parameters by grouping them into a meaningful parameter. These clusters display a proportional measurement of dimension parameters [24], [25]. For example, ρ is the predator-to-prey growth rate ratio. When $\rho > 1$ or $\rho < 1$, then the condition shows significant ecological meaning. It means that prey can reproduce more rapidly than predator.

Positivity, Boundedness, and Permanence

This section consists the analysis results from model (3) which consists of positivity, boundedness, and permanence. Boundedness and permanency are demonstrated using the same methodology as in [26]. First, we introduce several lemmas.

Lemma 1. If $a, b > 0$ and $\frac{dX}{dt} \leq (\geq) X(t)\big(a - bX(t)\big)$ with $X(0) > 0$, then

$$\lim_{t\to\infty} \sup X(t) \leq \frac{a}{b}\Big(\lim_{t\to\infty} \inf X(t) \geq \frac{a}{b}\Big),$$

The following lemma is identical to lemma 1.

Lemma 2. If $a, b > 0$ and $\frac{dX}{dt} \leq X(t)\big(a - bX(t)\big)$ with $X(0) > 0$, then for all $t \geq 0$

$$X(t) \leq \frac{a}{b - Ce^{-at}} \text{ with } C = b - \frac{a}{X(0)}.$$

Thus, $X(t) \leq \max\left\{X(0), \frac{a}{b}\right\}$ for all $t \geq 0$.

Theorem 1. Model (3) with Eq. (4) has positive and bounded solutions for all $t \geq 0$.

Proof. a) In the model (3) for prey equation, an invariant set is $x = 0$. Thus, $x(t) > 0$ for all t with $x(0) > 0$. On the other hand, using the same way, from the predator equation of model (3), an invariant set is $y = 0$. Thus, $y(t) > 0$ for all t with $y(0) = 0$. It shows that all solutions can cross into boundary R_+^2, but never out of R_+^2.

b) Considering the positivity of x and y, the prey equation for model (3) becomes as follows.

$$\frac{dx}{dt} = x\left(1 - x - y - \frac{\alpha}{x + \beta}\right) \leq x(1 - x). \tag{5}$$

Lemma 2 allows us to obtain

$$x(t) \leq \max\{x(0), 1\} \equiv M_1 \ \forall \ t \geq 0.$$

Further, the predator equation of model (3) can be rewritten as follows.

$$\frac{dy}{dt} = \rho y\left(1 - \frac{\gamma y}{x}\right) - \frac{hy}{c + y} \leq \rho y\left(1 - \frac{\gamma y}{M_1}\right). \tag{6}$$

We obtain the following equation using **Lemma 2**.

$$y(t) \leq \max\left\{y(0), \frac{M_1}{\gamma}\right\} \equiv M_2 \; \forall \; t \geq 0.$$

This implies that all solutions are bounded. Consequently, the model (3) is dissipative.

To prove the permanence of model (3), we recall the definition as in [27].

Definition 1. Model (3) is permanence if ϑ_1 and ϑ_2 are positive constants with $0 < \vartheta_1 < \vartheta_2$. Thus, model (3) with Eq. (4) fulfill

$$\min\left\{\lim_{t\to\infty} \inf X(t), \lim_{t\to\infty} \inf Y(t)\right\} \geq \vartheta_1,$$
$$\max\left\{\lim_{t\to\infty} \sup X(t), \lim_{t\to\infty} \sup Y(t)\right\} \leq \vartheta_2.$$

Theorem 2. There is permanency in model (3) with Eq. (4) when $\frac{1}{\gamma} + \frac{\alpha}{\beta} < 1$ and $c\rho > h$.

Proof. Using **Lemma 1**, it is obtained a condition from equation (5), i.e. $0 < x(t) < 1 \; \forall \; t \leq T$. Meanwhile, from equation (6), it is obtained $y(t) \leq \frac{1}{\gamma} \; \forall \; t \leq T$. Therefore, from prey equation, we have

$$\frac{dx}{dt} = x\left(1 - x - y - \frac{\alpha}{x+\beta}\right) \geq x\left(1 - x - \frac{1}{\gamma} - \frac{\alpha}{\beta}\right) = x(\xi_1 - x) \; \forall \; t \leq T,$$

where $\xi_1 = 1 - \frac{1}{\gamma} - \frac{\alpha}{\beta}$. If $\xi_1 > 0$, then $\frac{1}{\gamma} + \frac{\alpha}{\beta} < 1$. Further, from **Lemma 1**, we have

$$\lim_{t\to\infty} \inf x(t) \geq \xi_1.$$

From the predator equation, we obtain

$$\frac{dy}{dt} = y\left(\rho - \frac{\rho\gamma y}{x} - \frac{h}{c+y}\right) \geq y\left(\left(\frac{c\rho - h}{c}\right) - \rho\gamma y\right) = y(\xi_2 - \rho\gamma y) \; \forall \; t \leq T,$$

where $\xi_2 = \frac{c\rho - h}{c}$. If $\xi_2 > 0$, then $c\rho > h$. Thus, using **Lemma 1**, we have

$$\lim_{t\to\infty} \inf y(t) \geq \xi_2,$$

Further, from equation (5) and (6), we analyze it using **Lemma 1**. Thus, we obtain

$$\lim_{t\to\infty} \sup x(t) \leq 1 \text{ and } \lim_{t\to\infty} \sup x(t)\, y(t) \leq \frac{M_1}{\gamma}.$$

Based analysis results above, we chose $\vartheta_1 = \min(\xi_1, \xi_2)$ and $\vartheta_2 = \max\left(1, \frac{M_1}{\gamma}\right)$. Finally, we can conclude that model (3) is permanence.

Equilibrium and Existence

Let $\frac{dx}{dt} = \frac{dy}{dt} = 0$, then it is obtained the equilibrium points as follows.

a) The origin point $E_1(0,0)$ is always exist.

b) The predator free points are $E_2(x_L, 0)$ and $E_3(x_H, 0)$, where x_L and x_H are obtained from

$$x^2 + (\beta - 1)x + (\alpha - \beta) = 0,$$

with

$$x_L = \frac{1-\beta}{2} + \sqrt{\left(\frac{\beta+1}{2}\right)^2 - \alpha} \text{ and } x_H = \frac{1-\beta}{2} - \sqrt{\left(\frac{\beta+1}{2}\right)^2 - \alpha}\,.$$

These points exist when $\left(\frac{\beta+1}{2}\right)^2 > \alpha. E_2(x_L, 0)$ exists when the Allee's effect is small. Meanwhile, $E_2(x_L, 0)$ and $E_3(x_H, 0)$ exist when the Allee's effect is enormous and $0 < \beta < 1$,

c) The interior point is $E_4(x^*, y^*)$ with $x^* = \frac{\rho\gamma y^*(c+y^*)}{\rho(c+y^*)-h}$ and y^* is the root of polynomial equation as follows.

$$A_1(y^*)^4 + A_2(y^*)^3 + A_3(y^*)^2 + A_4 y^* + A_5 = 0, \tag{5}$$

with

$$\begin{aligned}
A_1 &= \gamma\rho^2(\gamma + 1),\\
A_2 &= \rho^2(\gamma + 1)(2c\gamma + \beta) - \rho\gamma(h + \rho),\\
A_3 &= \rho\big(c\rho(c\gamma + 2\beta)(\gamma + 1) - \beta h(\gamma + 2) + \rho(\alpha - \beta) - c\gamma(h + \rho) - \gamma(c\rho - h)\big),\\
A_4 &= (c\rho - h)(\beta c\rho(\gamma + 1) + 2\rho(\alpha - \beta) - c\gamma\rho - \beta h),\\
A_5 &= (c\rho - h)^2(\alpha - \beta).
\end{aligned}$$

Notice that A_1 is always positive. The Allee's small effect makes $A_5 < 0$. Meanwhile, the Allee's high effect makes $A_5 > 0$. Descartes Rule of Sign is applied to obtain how many positive real roots of equation (5) are possible. Thus, the maximum number of positive real roots are at most three positive real roots when $A_5 < 0$, and four positive real roots when $A_5 > 0$. The possibilities can be summarized in **Table 1**. The point $E_4(x^*, y^*)$ exists when $\rho(c + y^*) > h$.

Table 1. The Quantity of Potentially Positive Actual Roots

For $A_5 < 0$								
	$A_1 > 0$							
	$A_2 < 0$				$A_2 > 0$			
	$A_3 > 0$	$A_3 < 0$	$A_3 > 0$	$A_3 < 0$	$A_3 > 0$	$A_3 < 0$	$A_3 > 0$	$A_3 < 0$
	$A_4 > 0$	$A_4 > 0$	$A_4 < 0$	$A_4 < 0$	$A_4 > 0$	$A_4 > 0$	$A_4 < 0$	$A_4 < 0$
y^*	3	3	3	1	1	3	1	1
For $A_5 > 0$								
	$A_1 > 0$							
	$A_2 < 0$				$A_2 > 0$			
	$A_3 > 0$	$A_3 < 0$	$A_3 > 0$	$A_3 < 0$	$A_3 > 0$	$A_3 < 0$	$A_3 > 0$	$A_3 < 0$
	$A_4 > 0$	$A_4 > 0$	$A_4 < 0$	$A_4 > 0$	$A_4 > 0$	$A_4 > 0$	$A_4 < 0$	$A_4 < 0$
y^*	2	2	4	2	0	2	2	2

The Stability of Equilibrium Point

We examine the stability of equilibrium points in this section. The analytical stability is split into local and global stability for both strong and weak Allee effects.

The Local Stability

The equilibrium points' local stability for the strong and weak Allee effects are shown in this section. We examine the dynamics near equilibrium points by taking the following Jacobian matrix (see **Equation 6**). From equation 6, we obtain eigenvalues (λ) that can be used determined the stability of equilibrium points. Here are some theorems for each point.

$$J(E)=\begin{bmatrix} 1-2x-\dfrac{\alpha\beta}{(x+\beta)^2}-y & -x \\ \dfrac{\rho\gamma y^2}{x^2} & \rho-\dfrac{2\rho\gamma y}{x}-\dfrac{hc}{(c+y)^2} \end{bmatrix}. \tag{6}$$

Theorem 3. The Allee's high effect causes point $E_1(0,0)$ to become asymptotic local stability. But, the Allee's small effect causes point E_1 to become a saddle point.

Proof. In this part, we cannot determine the Jacobian matrix for point E_1 because it is found an undefined term $\frac{y}{x}$. Therefore, the blow-up method as in [23] is used to investigate the stability of point E_1. Let $x = x$ and $y = xu$, then model (3) become

$$\begin{aligned} \frac{dx}{dt} &= x\left(1-x-\frac{x}{x+\beta}-xu\right), \\ \frac{dy}{dt} &= u\left(\rho-\rho\gamma u-\frac{h}{c+xu}-1+x+\frac{\alpha}{x+\beta}+xu\right). \end{aligned} \tag{7}$$

From model (7), it can be obtained two points, namely $\hat{E}_1(0,0)$ and $\hat{E}_2(0,\hat{u})$ where $\hat{u} = \frac{\beta(c\rho-h)+c(\alpha-\beta)}{\rho\gamma\beta c}$. The point $\hat{E}_2$ exists if $c\rho > h$ and $\alpha > \beta$ and the point $\hat{E}_1$ is always exist.

Now, we investigate the stability of these points. First, we must find the Jacobian matrix at point $\hat{E}_1$. The Jacobian matrix for model (7) at point $\hat{E}_1$ is presented as follows.

$$J(\hat{E}_1)=\begin{bmatrix}\dfrac{\beta-\alpha}{\beta} & 0\\ 0 & \dfrac{(c\rho-h)}{c}+\dfrac{(\alpha-\beta)}{\beta}\end{bmatrix}. \tag{8}$$

By considering the existence condition of $\hat{E}_2$, it is confirmed from equation (8) that the unstable point $\hat{E}_1$ is caused by the Allee's small effect because both eigenvalues are positive, that is $\lambda_{1,2}>0$. But, the Allee's high effect makes point $\hat{E}_1$ to become a saddle point because the first eigenvalue is negative ($\lambda_1<0$) and the second eigenvalue is positive ($\lambda_2>0$).

Next, to investigate the stability of point $\hat{E}_2(0,\hat{u})$, we must determine the Jacobian matrix at point $\hat{E}_2$, that is

$$J(\hat{E}_2)=\begin{bmatrix}\dfrac{\beta-\alpha}{\beta} & 0\\ \hat{u}\left(1+\hat{u}+\dfrac{h\hat{u}}{c^2}-\dfrac{\alpha}{\beta^2}\right) & -\left(\dfrac{(c\rho-h)}{c}+\dfrac{(\alpha-\beta)}{\beta}\right)\end{bmatrix}. \tag{9}$$

From equation (9), we can confirm that the point $\hat{E}_2$ doesn't exist due to the weak Allee. Meanwhile, the point $\hat{E}_2$ is point of asymptotic local stability because the strong Allee. It is caused by both negative eigenvalues, that is $\lambda_{1,2}<0$. Therefore, we can conclude that **Theorem 3** is proven.

Theorem 4. a) If we consider $c\rho>h$, then it is the weak Allee and the point $E_2(x_L,0)$ is a saddle point. But, the unstable point $E_2(x_L,0)$ and $E_3(x_H,0)$ are caused by the strong Allee. b) If we consider $c\rho<h$, then the weak Allee makes the point $E_2(x_L,0)$ as asymptotic local stability point. But, the strong Allee makes $E_2(x_L,0)$ as asymptotic local stability point and $E_3(x_H,0)$ as saddle point.

Proof. It is known the Jacobian matrix.

$$J(E_{2,3})=\begin{bmatrix}\dfrac{\alpha x_{L,H}}{\left(x_{L,H}+\beta\right)^2}-x_{L,H} & -x_{L,H}\\ 0 & \dfrac{c\rho-h}{c}\end{bmatrix}. \tag{10}$$

From equation (10), we get the Eigen value, that is $\lambda_1=\frac{\alpha x_{L,H}}{\left(x_{L,H}+\beta\right)^2}-x_{L,H}$ and $\lambda_2=\frac{c\rho-h}{c}$. Based on the condition of permanence, it is clear that the second eigenvalue λ_2 is always positive. Therefore, the first eigenvalue λ_1 at $E_{2,3}$ affects the stability. E_2 is a saddle point when there is small Allee effect. This is caused by the first eigenvalue is negative, that is $\lambda_1<0$. But, when there is strong Allee effect, then E_2 and E_3 are unstable because the first eigenvalue λ_1 for both point is always positive. Therefore, **Theorem 4(a)** is proven.

Now, we consider $c\rho<h$. This means that the harvest variables greatly affect the interaction of prey and predators. It can show that the second eigenvalue λ_2 is always negative. Thus, the small Allee

effect makes E_2 to become an asymptotic local stability point because both negative eigenvalues $\lambda_{1,2} < 0$. Furthermore, the high Allee effect makes E_2 to become asymptotic local stability point with both negative eigenvalues $(\lambda_{1,2} < 0)$ and E_3 becomes saddle point when the first eigenvalue is positive and the second eigenvalue is negative, that is $\lambda_1 > 0$ and $\lambda_2 < 0$. Thus, **Theorem 4(b)** is proven.

From Theorem 3 and 4, we get Lemma 3 as follows.

Lemma 3. By using $c\rho > h$, it is obtained that the small Allee effect can cause E_1 and E_2 together to constitute the saddle point. But, the high Allee effect can make E_1 to become local stability point and the points E_2 and E_3 become unstable point.

Theorem 5. Let

$$\begin{aligned}
\varphi_1 &= \left(\frac{\alpha x^*}{(x^*+\beta)^2}\right)\left(\frac{hy^*}{(c+y^*)^2}\right) + \frac{\rho\gamma x^* y^*}{x^*} + \frac{\rho\gamma (y^*)^2}{x^*},\\
\varphi_2 &= \frac{\alpha\rho\gamma x^* y^*}{x^*(x^*+\beta)} + \frac{hx^*y^*}{(c+y^*)^2},\\
\varphi_3 &= \frac{\alpha x^*}{(x^*+\beta)^2} + \frac{hy^*}{(c+y^*)^2},\\
\varphi_4 &= x^* + \frac{\rho\gamma y^*}{x^*}.
\end{aligned}$$

All points of $E_4(x^*, y^*)$ are asymptotic local stability when these points satisfy several conditions, that is $\varphi_1 > \varphi_2$ and $\varphi_3 < \varphi_4$.

Proof. The Jacobian matrix at E_4 can be identified as follows.

$$J(E_4) = \begin{bmatrix} x^*\left(\frac{\alpha}{(x^*+\beta)^2} - 1\right) & -x^* \\ \frac{\rho\gamma (y^*)^2}{(x^*)^2} & y^*\left(\frac{h}{(c+y^*)^2} - \frac{\rho\gamma}{x^*}\right) \end{bmatrix}. \tag{11}$$

Furthermore, we obtain the trace and determinant of equation (11), that is

$$\begin{aligned}
\det\big(J(E_4)\big) &= \left(\frac{\alpha x^*}{(x^*+\beta)^2} - x^*\right)\left(\frac{hy^*}{(c+y^*)^2} - \frac{\rho\gamma y^*}{x^*}\right) + \frac{\rho\gamma (y^*)^2}{x^*},\\
&= \left(\left(\frac{\alpha x^*}{(x^*+\beta)^2}\right)\left(\frac{hy^*}{(c+y^*)^2}\right) + \frac{\rho\gamma x^* y^*}{x^*} + \frac{\rho\gamma (y^*)^2}{x^*}\right) - \left(\frac{\alpha\rho\gamma x^* y^*}{x^*(x^*+\beta)} + \frac{hx^*y^*}{(c+y^*)^2}\right),\\
\mathrm{tr}\big(J(E_4)\big) &= \left(\frac{\alpha x^*}{(x^*+\beta)^2} + \frac{hy^*}{(c+y^*)^2}\right) - \left(x^* + \frac{\rho\gamma y^*}{x^*}\right).
\end{aligned}$$

By applying the Routh-Hurwitz criteria, it is obtained that these points of E_4 are asymptotic local stability if $\det\big(J(E_4)\big) > 0$ and $\mathrm{tr}\big(J(E_4)\big) < 0$. Therefore, we get several conditions as in **Theorem 5** that indicate the stability of E_4. Finally, **Theorem 5** is proven.

The Global Stability

This section looks into the global stability using the Lyapunov function. We have observed that the global stability is only caused at E_4. Therefore, **Theorem 6** is given as follows.

Theorem 6. All points of E_4 are asymptotic global stability when these points satisfy some conditions, that is $\bar{A} > \alpha$ and $\frac{\rho\gamma}{x^*} > \frac{h}{\bar{B}}$ with $\bar{A} = \beta(x^* + \beta)$ and $\bar{B} = c(c + y^*)$.

Proof. First, we consider the Lyapunov function as follows.

$$V(x,y) = v_1\left((x - x^*) - x^* \log\left(\frac{x}{x^*}\right)\right) + v_2\left((y - y^*) - y^* \log\left(\frac{y}{y^*}\right)\right).$$

At the point E_4, it is clear that $V(E_4) = 0$. Meanwhile, the time derivative V is presented as follows.

$$\begin{aligned}\frac{dV}{dt} &= v_1 \frac{(x - x^*)}{x}\frac{dx}{dt} + v_2 \frac{(y - y^*)}{y}\frac{dy}{dt},\\ &= -\left(v_1\left(\frac{A - \alpha}{A}\right)(x - x^*)^2 + v_2\left(\frac{\rho\gamma}{x^*} - \frac{h}{B}\right)(y - y^*)^2 + \left(v_1 - v_2\frac{\rho\gamma y}{xx^*}\right)(x - x^*)(y - y^*)\right),\end{aligned}$$

where $A = (x + \beta)(x^* + \beta)$ and $B = (c + y)(c + y^*)$. Furthermore, we take $v_1 = \frac{\rho\gamma y}{xx^*}$ and $v_2 = 1$. Thus, we obtain

$$\frac{dV}{dt} = -\left(\frac{\rho\gamma y}{xx^*}\left(\frac{\bar{A} - \alpha}{\bar{A}}\right)(x - x^*)^2 + \left(\frac{\rho\gamma}{x^*} - \frac{h}{\bar{B}}\right)(y - y^*)^2\right) < 0.$$

Therefore, $\frac{dV}{dt} < 0$ when $\bar{A} > \alpha$ and $\frac{\rho\gamma}{x^*} > \frac{h}{\bar{B}}$ with $\bar{A} = \beta(x^* + \beta)$ and $\bar{B} = c(c + y^*)$. Thus, **Theorem 6** is proven.

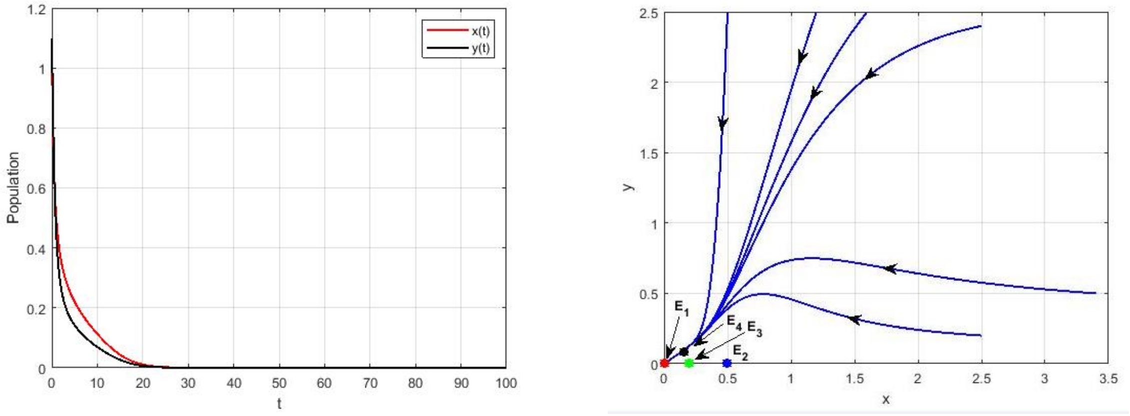
Numerical Simulation

Using the predictor-corrector approach, In order to numerically observe the dynamics of model (3), we analyze a number of numerical experiments. Because there is a lack of field data, we utilize the fictitious parameter values in the simulation (see **Table 2**). The parameter from model (3) that is very influential is the Allee effect parameter. Therefore, the simulation for the Allee effect and the simulation for the harvest are the two simulations that make up this one.

In this section, the numerical simulations are divided 5 simulations. Simulation 1 is conducted to determine the impact of the strong Allee effect, where $\alpha = 0.4 > \beta = 0.3$. Simulation 2 and 3 are conducted to determine the impact of the weak Allee effect, where $\alpha = 0.4 < \beta = 0.48; 1.3$. Simulation 4 is conducted to find the effect of the harvest variable when the Allee effect is weak. Finally, to find out how much influence the Allee effect and the harvest variable have, simulation 5 is conducted with the selected parameter being the parameters β and h. Therefore, the parameters β and h are set to value that are always changing.

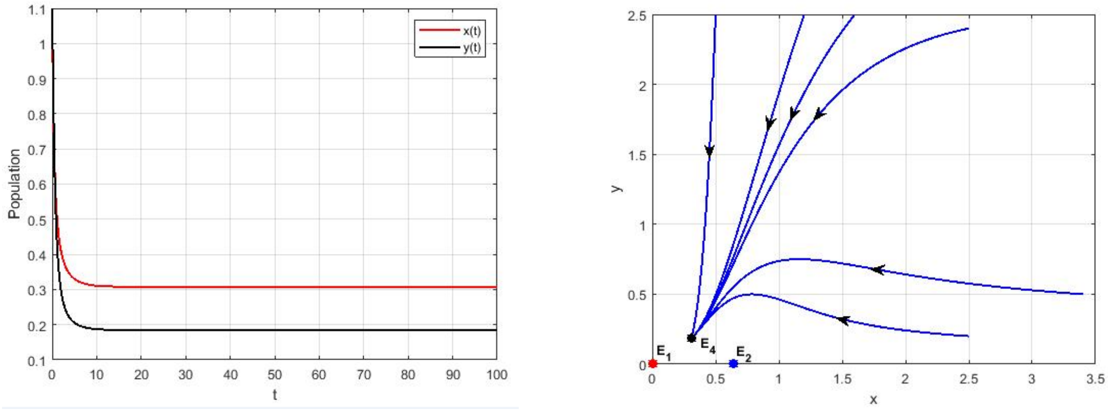
Table 2. Parameters for Numerical Simulations

Parameters	**Numerical Simulations**			
	I	**II**	**III**	**IV**
α	0.4	0.4	0.4	0.4
β	0.3	0.48	1.3	1.3
γ	1.5	1.5	1.5	1.5
ρ	1.75	1.75	1.75	1.75
c	0.1	0.1	0.1	0.1
h	0.05	0.05	0.05	1.05



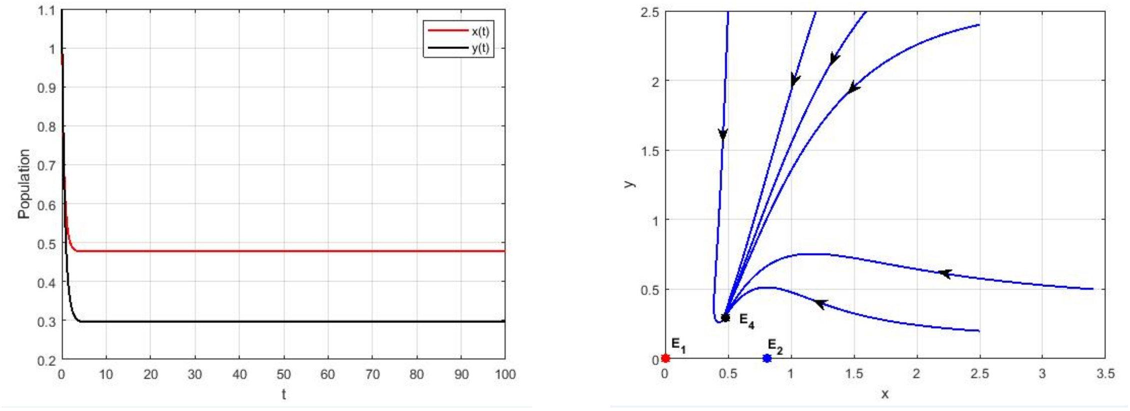
(a) (b)

Figure 1. Simulation 1: (a) Time Series with $t = 100$ Depicting Extinction of All Populations When the Allee Effect, (b) Phase Portrait Showing All Solutions Converge to E_1 with $\beta = 0.3$



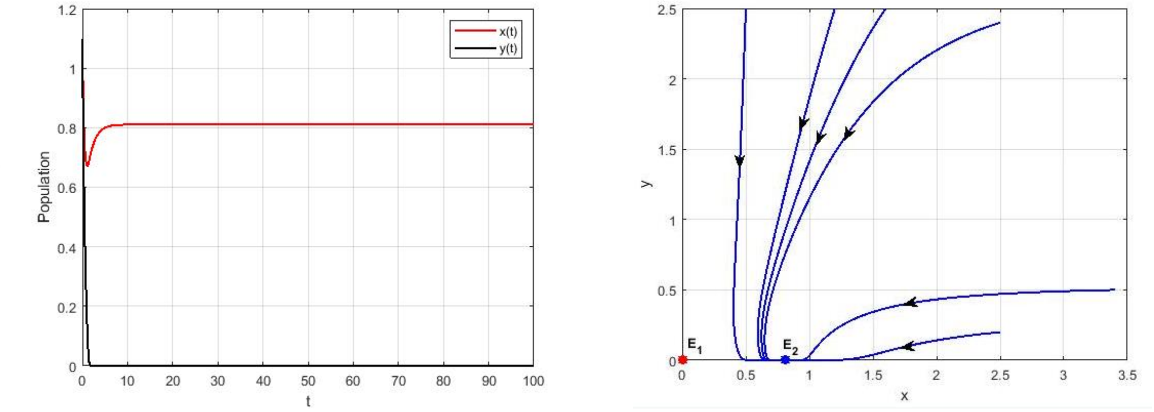
(a) (b)

Figure 2. Simulation 2: (a) Time Series with $t = 100$ Depicting Existence of All Populations When the Allee Effect is Weak, (b) Phase Portrait Showing All Solutions Converge to E_4 with $\beta = 0.48$



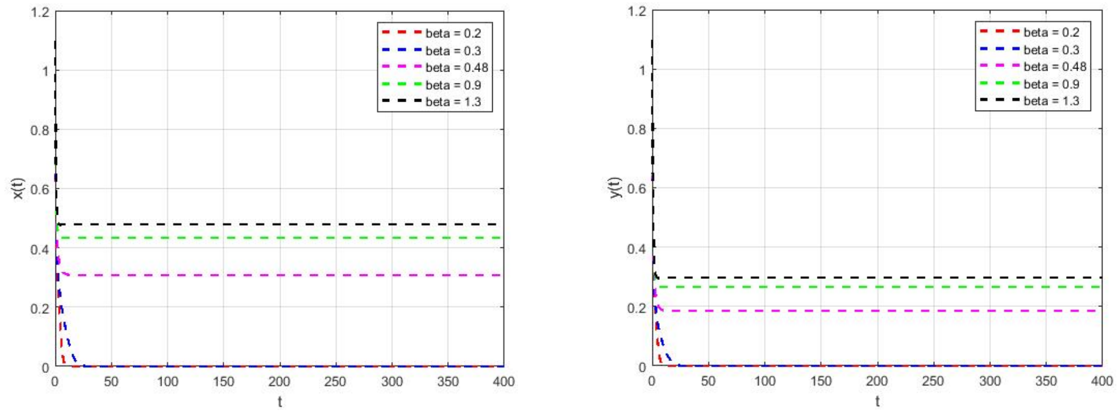
(a) (b)

Figure 3. Simulation 3: (a) Time Series with $t = 100$ Depicting Existence of All Populations When the Allee Effect is Weak, (b) Phase Portrait Showing All Solutions Converge to E_4 with $\beta = 1.3$

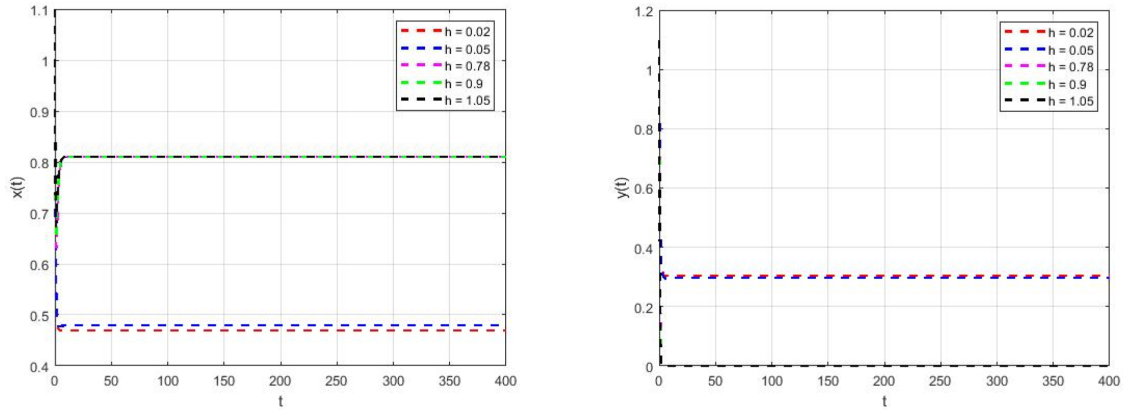


(a) (b)

Figure 4. Simulation 4: (a) Time Series with $t = 100$ Depicting Existence of All Populations When the Allee Effect is Weak and the Harvest Rate is High, (b) Phase Portrait Showing All Solutions Converge to E_4 with $\beta = 1.3$ and $h = 1.05$



(a) (b)



(c) (d)

Figure 5. Time Series for Simulation 5 Using the Parameter of Simulation 1: Simulation (a) and (b) Depicting the Allee Effect with β Always Changing; Simulation (c) and (d) Depicting the Harvest Effect with h Always Changing

From the results of analysis, the Allee effect determines whether equilibrium points are stable. As observed in simulation 1, all solutions tend to point E_1 when the Allee effect is substantial (see **Figure 1b**). This indicates that the strong Allee effect can cause prey and predator populations to experience extinction (see **Figure 1a**).

Based on the numerical results, it is obtained that $\frac{1}{\gamma}+\frac{\alpha}{\beta}=2>1$ dan $c\rho-h=0.125>0$. In this condition, the permanence conditions of model (3) are not satisfied, while all points (E_1, E_2, E_3, E_4) exist. If we look at all solutions that occur, the solutions converge to point E_1 (see **Figure 1b**). This strengthens the statement above that both populations of prey and predators cannot live together and they experience extinction. Because of the significant Allee effect, the prey population goes extinct, depriving the predator population of a food source and ultimately leading to their own extinction.

The Allee effect continues to affect how the two populations interact in simulation 2. In these circumstances, it is evident that every solution tends to point E_4 (see **Figure 2b**). This implies that populations of prey and predators do not go extinct as the Allee effect weakens (see **Figure 2a**).

However, if we analyze the permanence condition, then we obtain that $\frac{1}{\gamma}+\frac{\alpha}{\beta}=1.5>1$ and $c\rho-h=0.125>0$. Meanwhile, the existing equilibrium points are E_1, E_2, and E_4. Apart from that, the stability in simulation 2 is only local, not global. This is proven from theorem 6 which is not satisfied. Meanwhile, **Lemma 3** and **Theorem 4** are satisfied. This means that the two populations cannot actually live together in an ecosystem. The factor that causes this condition is the Allee effect. This effect still has a role in an ecosystem so that the continuity of two populations cannot live together or there is still a possibility that both will become extinct.

Just like in simulation 2, simulation 3 still considers the Allee effect in the interaction of the two populations. In this simulation, all solutions lead to point E_4 as shown in **Figure 3b**. Both populations survive and do not become extinct (see **Figure 3a**). These conditions are the same as those that occurred in simulation 2.

However, there are differences between simulation 2 and simulation 3 if we look at the numerical results. Based on the data obtained, the condition $\frac{1}{\gamma}+\frac{\alpha}{\beta}=0.973<1$ dan $c\rho-h=0.125>0$. The existing equilibrium points are E_1, E_2, and E_4. In addition, **Theorem 4**, **Lemma 3**, and **Theorem 6** are satisfied in this simulation. This means that model (3) satisfies the permanence condition the stability of E_4 is both local and global. From these results, it can be shown that the two populations are not extinct and they can live together in one ecosystem for long time.

An interesting thing is shown when the value of the harvest variable is increased to $h = 1.05$ as in simulation 4. Under this condition, all solutions converge to point E_2 (see **Figure 4b**). In addition, the predator population is headed for extinction, while the prey population is not extinct (see **Figure 4a**). This means that predators cannot maintain their species due to the effect of humans who hunt predators, while the prey population is successful in maintaining its species.

Based on the numerical results, it is obtained that $\frac{1}{\gamma} + \frac{\alpha}{\beta} = 0.974 < 1$ and $c\rho - h = -0.875 < 0$. Meanwhile, the existing equilibrium points are E_1 and E_2. Furthermore, **Theorem 4** is satisfied. This means that model (3) does not satisfy the permanence condition and the point E_2 is only locally stable. These results strengthen previous statements that the harvest variable causes predator to become extinct. Meanwhile, the prey population can survive because there are no predator to prey on it.

Based on the numerical results above, the Allee effect and harvest are variables that greatly affect the interaction between prey and predator. Therefore, the next experiment is carried out to see how high the impact of the Allee effect and harvest with several numerical simulations. The variable β is chosen to see the impact of the Allee effect, while the variable h is chosen as a benchmark to see the effect of harvest. The numerical simulation results can be seen in **Figure 5** for each variable β and variable h.

To observe the effect of Allee, we use several parameters as in simulation 1, but variable β is created to always change. The simulation results are shown in **Figure 5a** and **Figure 5b**. Under the **Figure 5a** and **Figure 5b**, it can be seen that when we choose $\alpha = 0.4$ and the value of β changes from 0.2 to 1.3, then Allee effect changes from strong to weak. This change causes the prey and predator populations to become extinct together when $\beta = 0.2; 0.3$, while the prey and predator populations can live together when $\beta = 0.48; 0.9; 1.3$. This means that a strong Allee effect causes both populations to become extinct, while a weak Allee effect causes both populations to survive.

To observe the effect of harvest variable h, some parameters in simulation 4 are reused in this simulation. However, the variable h is changed to a different values. From the simulation carried out, numerical results are obtained as in **Figure 5c** and **Figure 5d**. Under **Figure 5c** and **Figure 5d**, it can be seen that the value h is increased from 0.02 to 1.05. Changes in the value h can cause changes in the number of prey and predator populations. When the value h is increased, then the number of prey increases and even becomes constant in $x = 0.8105$, while the number of predator decreases and even becomes extinct. This is caused be frequent hunting carried out by humans so that the predator population becomes extinct. Meanwhile, the prey population can survive because there are no predators in the ecosystem. Thus, the existence of prey population is not disturbed even if the harvest variable is high and the Allee effect becomes weak or strong.

Conclusion

When calculating the interactions between populations of sharks and Atlantic cod, the Leslie-Gower model has been taken into consideration. By involving Allee and harvest effect, it is shown that the equilibrium point that appears is $E_1(0,0), E_2(x_L, 0), E_3(x_H, 0),$ and $E_4(x^*, y^*)$, where each point has certain condition of existence. In addition, the condition of permanence, boundedness, and positivity are also considered to guarantee the analysis results. Stability analysis is divided into 2, namely local and global. In this case, all equilibrium points can be locally stable, while global stability is only found at point E_4. Numerical simulations are run at the conclusion of the session to validate the analysis findings.

Based on the simulation results, it is found that the Allee effect and harvest are variables that effect the number of the two populations. In this condition, the condition of permanence is used to ensure that two populations live together. Both populations become unviable or become extinct when the Allee effect intensifies. It is possible for the two populations to coexist in the same ecosystem when the Allee effect is weak, as long as the permanence condition is satisfied. Only the prey population lives when the harvest variable rises, however, and the predator population goes extinct.

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